.

Physics 209: Notes on Special Relativity

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1. The Principle of Relativity

The principle of relativity is an example of an *invariance law*. Here are several such laws:

All other things being the same:

1. It doesn't matter where you are. (*Principle of translational invariance in space.*)

2. It doesn't matter when you are. (*Principle of translational invariance in time.*)

3. It doesn't matter what direction you are oriented in. (*Principle of rotational invariance.*)

4. If you're moving with fixed speed along a straight line, it doesn't matter how fast you're going. (*Principle of Relativity.*)

"It doesn't matter" means "the rules are the same". For example the law describing the force of gravity between two chunks of matter is the same whether they are in this Galaxy or another (principle of translational invariance in space). It is also the same today as it was a million years ago (principle of translational invariance in time). The law does not work differently depending on whether one chunk is east or north of the other one (principle of rotational invariance). Nor does the law have to be changed depending on whether you measure the force between the two chunks in a railroad station, or do the same experiment with the two chunks on a uniformly moving train (principle of relativity).

"All other things being the same" raises deep questions. In the case of translational invariance it means that when you move the experiment to a new place or time you have to move everything relevant; in the case of rotational invariance you have to turn everything relevant. In the case of the principle of relativity you have to set everything relevant into motion. If everything relevant turned out to be the entire universe you might wonder whether there was any content to the principle.

One can thus immediately descend into deep philosophical questions from which some never reappear. We shall not do this. We are interested in how such principles work on the practical level, where they are usually unproblematic. You easily can state a small number of relevant things that have to be the same and that is quite enough. When the principle doesn't work, invariably you discover that you have overlooked something simple that is relevant. Not only does that fix things up, but often you learn something new about nature that proves useful in many entirely different contexts. For example if the stillness of the air was important for the experiment you did in the railroad station, then you had better be sure that when you do the experiment on a uniformly moving train you do not do it on an open flat car, where there is a wind, so all other relevant things are not the same, but in an enclosed car with the windows shut. If you hadn't realized that the stillness of the air was important in the station, then the apparent failure of the experiment to work the same way on the open flatcar would teach you that it was.

Invariance principles are useful because they permit us to extend our knowledge, and it is on that quite practical level that we shall be interested in the principle of relativity. It tells us that we can't distinguish between a state of rest and a state of uniform motion. All experiments give the same result, whether they are performed in a laboratory at rest or in a uniformly moving laboratory. It is important both to understand what the principle asserts and to know how to use it to extend knowledge.

On a deep level one can again get bogged down in subtle questions. What do we mean by rest or by uniform motion? We will again take a practical view. Uniform motion means moving with a fixed speed in a fixed direction.[1] Note that fixed direction is as important as fixed speed: a particle moving with fixed speed on a circle is not moving uniformly. You can clearly tell the difference between being in a plane moving at uniform velocity and being in a plane moving in turbulent air; between being in a car moving at uniform velocity and one that is accelerating or cutting a sharp curve or on a bumpy road or screeching to a halt. But you cannot tell the difference (without looking out the window) between being on a plane flying smoothly through the air at 400 miles per hour and being on a plane that is stationary on the ground.

In working with the principle of relativity the term *frame of reference* is extremely useful. A frame of reference (often simply called a "frame") is the (uniformly moving) system in terms of which you have chosen to describe things. For example a cabin attendant walks toward the front of the airplane at 2 mph in the frame of reference of the airplane. You start at the rear of the plane and want to catch up with him so you walk at 4 mph. If the plane is going at 500 mph then in the frame of reference of the ground this would be described by saying that the cabin attendant was moving forward at 502 mph, and you caught up by increasing your speed from 500 mph to 504 mph. One of the many remarkable things about relativity is how much one can learn from considerations of this apparently banal variety.

Another important term is *inertial frame of reference.* "Inertial" means stationary or uniformly moving. A rotating frame of reference is not inertial. Nor is one that oscillates back and forth. We will almost always be interested only in inertial frames of reference and will omit the term "inertial" except when we wish to contrast uniformly moving frames of reference to frames that move nonuniformly.

How do you know that a frame of reference is inertial? This is just another way of posing the deep question of how you know motion is uniform. It would appear that you have to be given at least one inertial frame of reference to begin with, since otherwise you could ask "Moving uniformly with respect to what?" Thus if we know that the frame in which a railroad station stands still is an inertial frame, then the frame of any train moving uniformly through the station is also an inertial frame. But how do we know that the frame of reference of the station is inertial?

[1] More compactly, moving with a fixed velocity. Note that the term "velocity" embraces both speed and direction of motion. Two boats moving 10 miles per hour, one going north and the other east, have the same speed but different velocities. Note also the extremely useful (as we shall see) convention that a *negative* velocity in a given direction means exactly the same thing as the corresponding positive velocity in the opposite direction: -5 mph east is exactly the same as 5 mph west.

Fortunately there is a simple physical test for whether a frame is inertial. In an inertial frame stationary objects on which no forces act remain stationary. It is this failure of a stationary object (you) to remain stationary (you are thrown about in your seat) that lets you know when the plane or car you are riding in (and the frame of reference it defines) is moving uniformly and when it is not. In our cheerfully pragmatic spirit, we will set aside the deep question of how you can know that no forces act. We will be content to stick with our intuitive sense of when the motion of an airplane (train, car) is or is not capable of making us seasick.

When specifying a frame of reference you can sometimes fall into the following trap: suppose you have a ball that (in the frame of reference you are using) is stationary before 12 noon, moves to the right at 3 feet per second (fps) between 12 pm and 1 pm, and to the left at 4 fps after 1 pm. By "the frame of reference of X" (also called the *proper frame* of X) one means the frame in which X is stationary. Now there is no *inertial frame of reference* in which the ball is stationary throughout its history. If you want to identify an *inertial* frame of reference as "the frame of reference of the ball" you must be sure to specify whether you mean the inertial frame in which the ball was stationary before 12, or between 12 and 1, or after 1. There are three different inertial frames that (depending on the time) serve as the frame of reference of the ball. Similarly for the Cannonball Express, which constitutes one inertial frame of reference as it zooms along a straight track at 120 mph from Syracuse to Chicago, and quite another as it zooms along the same track at the same speed on the way back. The frame of reference of an airplane buffeted by high winds may never be inertial. Nor is the frame of reference of the Cannonball Express as it moves smoothly along a curved stretch of track.

Here is another, more subtle trap, that many people (including, I suspect, some physicists) fall into:[2] people sometimes take the principle of relativity to mean, loosely speaking, that the behavior of a uniformly moving object should not depend on how fast it is moving, or, to put it slightly differently, that motion with uniform velocity cannot affect any properties of an object.

This is wrong. The principle of relativity only requires that if an object has certain properties in a frame of reference in which the object is stationary, then if the same object moves uniformly, it will have the same properties *in a frame of reference that moves uniformly with it.* On the other hand the properties of an object moving uniformly past you can certainly differ from the properties the same object has when it is standing still in front of you. To take a trivial example, when the object moves past you it has a non-zero speed; when it is stationary with you its speed is zero.

A more striking example is provided by the so-called Doppler effect: If a yellow light moves away from you at an enormous speed the color you see changes from yellow to red; if it moves toward you at an enormous speed the color changes from yellow to blue. So the color of an object can depend on whether it is moving or at rest (and in what direction it

[2] I only became fully aware of this trap a few years ago, when reading about some celebrated (but fallacious) objections to relativity by a physicist named Dingle.

is moving). All the principle of relativity guarantees is that if a light is seen to be yellow when it is stationary, then when it moves with uniform velocity it will still be seen as yellow *by somebody who moves with that same velocity.*

What we shall be almost exclusively interested in are some simple practical applications of the principle of relativity. To apply the principle of relativity it is essential to acquire the ability to visualize something looks when viewed from different inertial frames of reference. A useful mental device for doing this is to examine how a single set of events would be described by various people moving past them in trains moving uniformly with different speeds.

We will be applying the principle of relativity to learn some quite extraordinary things by examining the same sets of events in different frames of reference. Some of the things we shall learn in this way are so surprising that they are hard to believe. It is therefore essential to begin by getting some practice using the principle of relativity to learn some things that we might not have known before, which are not so amazing. The general trick for doing so is this:

Take a situation which you don't fully understand. Examine it in a new frame of reference in which you do understand it. Then translate your understanding in the new frame of reference back into the language of the old one.

Here is a simple example. Newton's first law of motion states that in the absence of an external force a uniformly moving body continues to move uniformly. But this law follows from the principle of relativity and a very much simpler law. The simpler law merely states that in the absence of an external force a stationary body continues to remain stationary.

To see how the more general law is a consequence of the simpler one, suppose we only know that the simpler law is true. The principle of relativity tells us it must be true in all inertial frames of reference. If we want to learn about the subsequent behavior of a ball initially moving at 50 fps in the absence of an external force, all we have to do is find an inertial frame of reference in which we can apply the simpler law. The frame we need is clearly the one that moves at 50 fps in the same direction as the ball, since in that frame of reference the ball is stationary. Putting it more concretely, think of how the ball looks from a train moving at 50 fps alongside the ball. In the frame of reference of the train the ball is stationary and we can apply the law that in the absence of an external force a stationary body remains stationary. But anything that is stationary in the train frame moves at 50 fps in the frame of reference in which we originally posed the problem. We conclude that since the ball remains stationary in the train frame in the absence of an external force, in the original frame it will continue to move at 50 fps in the absence of an external force.

So starting with the fact that undisturbed stationary objects remain stationary, we have used the principle of relativity to establish the much more general fact that undisturbed uniformly moving objects continue to move with their original velocity.[3] If you

[3] At the risk of complicating something simple, I feel obliged to remark that in reaching

already knew Newton's first law you might not be impressed at this line of thought, so let's examine a case where what we learn may not be quite so familiar.

Suppose we have two identical perfectly elastic balls. Identical elastic balls have the property that when you shoot them directly at each other with the same speed, after the collision each bounces back in the direction it came from with the same speed it had before the collision. Question: What happens if one of the balls is at rest and you shoot the other one directly at it?

Traditionally such questions are answered by invoking the conservation of energy and momentum. If you know how to use such conservation laws, you should forget this for now. It is entertaining and instructive that the question can be answered using nothing but the principle of relativity. In learning how to use the principle in this way you will acquire a conceptual skill that will be essential in understanding everything that is to follow. My own feeling is that answering such questions using the principle of relativity provides a deeper insight than answering them by applying conservation laws.[4] Here's how to figure out what happens, using only the principle of relativity:

First draw a picture illustrating the rule you know: when the balls move at each other with equal speeds, they simply rebound with the same speeds. Then draw a picture of the new situation. For concreteness let's take the original speed of the moving ball to be 10 fps. (Once you get good at this business you can simply take it to be a general speed u.) We want to know what goes in the box with the question mark in it.

	Before Collision	**After Collision**
Case 1 (known):	$(X) \rightarrow \ \leftarrow (Y)$	$\leftarrow (X) \qquad (Y) \rightarrow$
Case 2 (unknown):	$(X) \rightarrow \ (Y)$ $10 \ fps$	?

To understand what happens in Case 2 hop onto a train moving to the right at 5 fps.[5]

this conclusion we have implicitly assumed that if an object is undisturbed in one inertial frame of reference then it is undisturbed in any other inertial frame of reference — i.e. that the condition of no force acting on an object is an *invariant* condition independent of the frame of reference in which the object is described. Since such forces can be associated with jet engines being on or off, springs being compressed or slack, etc., this is a reasonable assumption.

[4] We shall examine such conservation laws only at the end of our study of relativity.

[5] Figuring out which train to take is crucial. In the present case we have picked this particular train because it is the one in whose frame the balls are moving with equal and opposite velocities, as we now confirm. Often it is obvious in what frame the unknown situation becomes the known one. Sometimes you have to think about it a bit. At such

Because we are now on a train moving to the right at 5 fps, ball X is moving to the right at an additional 5 fps. Since ball Y was stationary before we boarded the train, in the train frame it is moving to the *left* at 5 fps. *Therefore in the frame of this particular train Case 2 (before) is an instance of Case 1 (before).* But the principle of relativity assures us that any experiment we do with the two elastic balls must have the same outcome in any inertial frame of reference. Since the two balls are moving at each other with the same speed in the train frame, after the collision they must bounce away from each other, each still moving at 5 fps in the train frame. Now all that remains is to translate that answer back to the original frame of reference (which it is convenient to call the track frame). Ball X moves to the left at 5 fps in the train frame, so it is stationary in the track frame. Ball Y moves to the right at 5 fps in the train frame so it is moving to the right at 10 fps in the track frame. Therefore the complete picture is:

	Before Collision	**After Collision**
Case 1 (known):	$(X) \rightarrow \ \leftarrow (Y)$	$\leftarrow (X) \quad (Y) \rightarrow$
Case 2 (unknown):	$(X) \rightarrow \ (Y)$ $10\ fps$	$(X) \ (Y) \rightarrow$ $10\ fps$

We have used the principle of relativity to learn something new about identical elastic balls: if one is at rest and the other bumps it head-on, then the moving one comes to a complete stop and the stationary one moves off with the velocity the formerly moving one originally had. This is a fact familiar to all players of billiards, but not many of them realize that it is simply a consequence of the much more obvious fact (less frequently encountered in billiards) that when two balls collide head–on with equal and opposite speeds each bounces back the way it came with its original speed.

As a test to make sure you really understood the above argument, here are two similar questions. They can be answered by a similar application of the principle of relativity. If you understood the argument about the elastic balls, then with a little thought you should be able to answer both of the questions that follow:[6]

(1) Two identical sticky balls, depicted in the figure that follows as (X) and (Y), have the property that if they are fired directly at one another with equal speeds, then they stick together upon collision and the resulting compound ball (XY) is stationary. If

times trial and error is a useful method. Ask yourself how the balls are described in a frame moving to the right at 1 fps, 2 fps, etc. Frequently the velocity you need then becomes evident.

[6] Conversely, if you don't see how to answer these questions after some thought, then you probably didn't really understand what I was saying about the elastic balls, and should think your way through that again.

a sticky ball is fired at 10 fps directly at another identical sticky ball that is stationary and the two stick together, with what speed and in what direction will the compound ball move after the collision?

	Before Collision	**After Collision**
Case 1 (known):	$(X) \rightarrow \; \leftarrow (Y)$	(XY)
Case 2 (unknown):	$(X) \rightarrow \quad (Y)$	?

(2). Suppose we have two elastic balls, but one of them (B) is very big and the other (s) is very small. If the big ball is stationary and the small ball is fired directly at it, the small ball simply bounces back in the direction it came from with the same speed, and the big ball stays at rest. With what speed will each ball move after the collision, if the small ball is stationary and the big ball is fired directly at it with a speed of 15 fps?

	Before Collision	**After Collision**
Case 1 (known):	$(s) \rightarrow \quad (B)$	$\leftarrow (s) \quad (B)$
Case 2 (unknown):	$(s) \quad \leftarrow (B)$	?

In all of these cases you are told how two balls behave under certain conditions and are asked what will happen under a set of conditions that does not fit into the scheme you've been told about. You do this by first finding a frame of reference in which the new conditions do reduce to the ones you've been told about, then applying the rule you know in that frame, and finally translating the result back into the language of the original frame.

2. Nonrelativistic Addition of Velocities

Let us look a little more carefully at the reasoning we used to solve the bouncing ball problem. In addition to using the principle of relativity we also made use of all of the following facts:

1a. If a ball moves down the track at 10 fps in the track frame and a train moves down the track at 5 fps in the track frame then the ball moves down the train at 5 fps in the train frame.

1b. If a ball is stationary in the track frame and the train moves down the track at 5 fps in the track frame then the ball moves up the train at 5 fps in the train frame (or, equivalently, down the train at -5 fps in the train frame).

2a. If a ball moves down the train at 5 fps in the train frame and the train moves down the track at 5 fps in the track frame then the ball moves down the track at 10 fps in the track frame.

2b. If a ball moves up the train at 5 fps in the train frame and the train moves down the track at 5 fps in the track frame then the ball is stationary in the track frame.

We used 1a and 1b to translate the unknown asymmetric collision in the track frame into the known symmetric situation in the train frame. Then we appealed to the principle of relativity, which assures us that the rule[1] about symmetric collisions is valid in any inertial frame of reference. Using this rule we were able to say what happened after the collision in the train frame. Finally we used 2a and 2b to translate the situation after the collision in the train frame back into track-frame language.

As far as we know today, the principle of relativity is indeed valid. But what about assumptions 1a, 1b, 2a, and 2b? How can they be justified?

They are all applications of the following rule, which is called the Nonrelativistic Velocity Addition Law:[2]

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[1] "If two identical elastic balls collide with equal and opposite velocites then after the collision each bounces back in the direction it came from with its original speed."

[2] "Nonrelativistic" is an unfortunate term, but everybody uses it and so shall we. It does *not* mean, as you might think, "in contradiction to the principle of relativity". Unfortunately the body of lore constructed by applying the principle of relativity to certain facts about the speed of light has come to be known as the "Theory of Relativity". The term "nonrelativistic" is invariably used to mean "the way we used to think things were before we learned about the theory of relativity". Since (as we shall see) things actually are pretty much the way we used to think they were before we learned about the theory of relativity provided all speeds of interest are much less than the speed of light, "nonrelativistic" as used today means precisely "valid to a high degree of accuracy when all speeds are small compared with the speed of light".

If A, B, and C all move with uniform velocity then

$$v_{AC} = v_{AB} + v_{BC}, \tag{2.1}$$

where v_{XY} means "the velocity of X with repect to Y" or, more awkwardly, but more precisely, "the velocity of X in the frame of reference in which Y is stationary."

Let's check this out. We need a convention on when the velocity is positive. We will take the velocity of something to be positive (in a given frame) if its motion is to the right (in that frame), and negative, if its motion is to the left. Thus an object going right at a speed of 5 fps has a velocity of 5 fps, but an object going left at a speed of 5 fps has a velocity of -5 fps.

Facts 1a and 2a are both instances of (2.1) with A being the ball, B being the train, and C being the track: the velocity of the ball in the track frame (10 fps) is the velocity of the ball in the train frame (5 fps) plus the velocity of the train in the track frame (5 fps).

More precisely, 2a says this directly, but 1a actually says that the velocity of the ball in the train frame (5 fps) is the velocity of the ball in the track frame (10 fps) *minus* the velocity of the train in the track frame:

$$v_{AB} = v_{AC} - v_{BC}, \tag{2.2}$$

which looks slightly different from (2.1). However it is an important fact[3] that

$$v_{XY} = -v_{YX}; \tag{2.3}$$

i.e. if X moves with a certain speed with respect to Y, then Y moves with that same speed with respect to X, but in the opposite direction. As an instance of this general fact we have $v_{CB} = -v_{BC}$, so (2.2) is equivalent to

$$v_{AB} = v_{AC} + v_{CB}. \tag{2.4}$$

But the relation (2.4) is just an instance of our original rule (2.1) (with the roles of B and C interchanged).

You should convince yourself that 1b and 2b are also instances of the general rules (2.1) and (2.3).

So what is the justification for rule (2.1)? Consider the instance of it provided by 2a. There we can justify rule (2.1) as follows: If the ball moves down the train[4] at 5 fps then

[3] And this fact remains valid even when speeds are comparable to that of light.

[4] I shall stop adding the cumbersome phrase "in the train frame" with the understanding then when we talk about the speed of the ball "down the train' we mean its speed in the train frame.

in one second the ball gets 5 feet further down the train. And if the train moves down the track[5] at 5 fps then in one second the train gets 5 feet further down the track. So in one second the ball gets 10 feet further down the track — the 5 it gains on the train and the additional 5 the train gains on the track. But the ball getting 10 feet further down the track in one second is precisely what we mean when we say the ball moves at 10 fps down the track. Who could doubt any of this?

Indeed, I encourage you not to doubt it until you have mastered the kinds of puzzles presented at the end of the preceding Chapter. Nevertheless, I call your attention to a dangerous phrase: "in one second". We have implicitly assumed that "in one second" means the same thing in the train frame as it does in the track frame. "Well," you will say, "of course it does. A second's a second." Suppose that's not true. Suppose "in a second" in the train frame means something different from "in a second" in the track frame. What happens to the argument we just gave? We would have to replace "in a second" by something like "in a second according to train-time" or "in a second according to track-time". The argument starts off fine, just a little more cumbersome:

> "If the ball moves down the train at 5 fps then in one second according to train time it gets 5 feet further down the train. And if the train moves down the track at 5 fps then in one second according to track time it gets 5 feet further down the track."

But then we come to:

> "So *in one second* the ball gets 10 feet further down the track — the 5 it gains on the train and the additional 5 the train gains on the track."

What can that italicized "in one second" mean here? The first 5 feet are gained in one second according to train time, the second 5 feet are gained in one second according to track time. Collapsing both into a single, unqualified "in one second" makes no sense. And indeed, when we get to the conclusion,

> "But the ball getting 10 feet further down the track in one second is precisely what we mean when we say the ball moves at 10 fps down the track."

we see that this only works if the italicized "in one second" means in one second according to track time, since what we precisely mean when we say the ball moves at 10 fps down the track is that it moves 10 feet in one second according to track time. So the conclusion rests on being able to replace "in one second according to train time" by "in one second according to track time".

For the moment we will not pursue this any further. But please be aware that the

[5] Similarly, by the speed of the train "down the track" we mean its speed in the track frame.

simple rule (2.1) telling us how velocities combine is based on the implicit assumption that there is nothing problematic about the idea of a single unique notion of time that can be used equally well in any frame of reference.[6]

[6] It was Einstein's great insight in 1905, that this apparently obvious assumption is, in fact, false.

3. The Speed of Light

When you turn on a light, how long does it take the light to get from the bulb to the things it illuminates? Galileo apparently tried to answer this by stationing two people with lanterns on top of distant mountains. Alice opens her lantern, Bob opens his the instant he sees Alice's, and Alice notes how much time passes between the moment when she turns hers on and the moment when she sees Bob's. Knowing that the peaks are a distance d apart, she just divides twice that distance by the delay time t and that's the speed of light:

$$c = 2d/t. \tag{1}$$

I don't know if Galileo worried about it, but there is a problem: how does Alice know how much of the delay is due to the time it took the light to get from her to Bob and back, and how much is due to the speed of Bob's response—i.e. the time it takes the reception of Alice's light at Bob's eyes to reach his brain and be converted into a signal that reaches the muscles in his arms that operate the tendons that cause his fingers to open his shutter.

There is an easy (but inspired) way to check this out. Simply do the experiment again with Alice on a second mountain farther away from Bob. Bob's response time won't change (assuming the light from Alice is not now too dim to see clearly) so the increase in the delay is entirely a result of the increase in the time it takes the light from the two lanterns to travel between the two mountains. Since this increase is just twice the increase in distance divided by the speed of light, Alice is back in a position to figure out the speed of light without having to know anything about Bob's response time. She simply uses (1) above with d being the *increase* in the distance between her and Bob in the two cases, and t being the *increase* in the time between her sending and receiving light signals.

But unfortunately, if she does this, Alice will observe no discernible change in the delay time. Either it takes no time at all for light to travel the extra distance (i.e. the speed of light is infinite) or Bob's sluggish response takes so very much longer than the light travel time that Alice simply can't tell the difference. The problem is that light travels so quickly that two terrestrial mountains within view of each other are much too close together for this method to work.

Three centuries later Galileo's unsuccessful attempt was realized by replacing the two mountains by the earth and the moon. The moon is so far away that it takes radar[7] over two seconds to get there and bounce back. But by then the speed of light was known to high precision by other methods.

To make Galileo's attempt work, either you have to increase the distance or be able to make much more accurate measurements of tiny intervals of time. The first successful

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[7] The speed of radar is the same as the speed of light. All electromagnetic radiation has the same speed in empty space.

estimate came from using astronomical distances. Careful observations of the eclipses of the moons of Jupiter (coincidentally discovered by Galileo) revealed that sometimes they lagged behind schedule by about ten minutes, and sometimes they came in ten minutes ahead. It was noted that they were ahead of schedule when the earth was closest to Jupiter, and behind when the earth was furthest away. This gave an estimate for the time it takes light to cross the orbit of the Earth: something like 20 minutes. This gives an estimate of some hundreds of thousands of miles per second for the speed of light. (Romer, 1676.)

In the 19th century a terrestrial measurement was done (Fizeau) by sharpening up the precision with which tiny time intervals could be measured. Imagine an axle with identical cog wheels at each end. Turn one of the wheels a little bit so that its teach come exactly in between the gaps in the teeth of the other wheel. Because of this misalignment, if you try to send a thin beam of light parallel to the axle through a gap between the teeth of one wheel, it will be blocked by a tooth of the second wheel. But if you now spin the whole thing extremely rapidly about the axle, you might hope that during the very tiny time it takes the light to pass between the two wheels, the second wheel will have turned just the tiny bit enough to allow a little bit of the light that passed through the gap in the first wheel to get through a small part of the gap in the second. After all, the wheels are spinning extremely fast and the teeth of the far away wheel have to move only a tiny fraction of a full turn to open up a passage for the light.

It turns out that for an axle short enough not to mess up this rather delicate alignment by a little bending, the light still travels too fast for this to work. However it is possible to introduce an enormous time-consuming detour for the light, in the form of a periscope–like perpendicular side journey with the help of four mirrors. When this was done, the sought for effect was observed, and the resulting estimate for the speed of light was in good agreement with that furnished by the earlier astronomical measurements.

Today we have highly sophisticated ways to measure the speed of light and know that it is 299,792,458 meters per second (m/s). Furthermore that is what it always shall be, because as of 1983 the meter has been *defined* to be not the distance between two scratches on a platinum iridium bar carefully kept in a vacuum in Paris, but as the distance light travels in 1/299,792,458 seconds.[8]

There are some useful coincidences associated with the speed of light being 299,792,458 meters per second:

1. The number is comfortably close to 300 million m/s (unless you require a precision of better than 0.1%) or 300,000 kilometers per second (km/s). Physicists are very used to

[8] The second is defined as the time it takes the light emitted by a certain atom—I forget which—under a particular set of circumstances—I forget what—to undergo a certain number of vibrations—I forget how many. The important point here is that our unit of length (the meter) is now no longer independent of our unit of time (the second).

taking it to be 3×10^8 m/s. So much so that there is a story that somebody once fouled up the report of a fine high precision experiment by using the number 3 rather than 2.9979 in converting the result into a more convenient form.

2. The corresponding English unit is about 186,000 miles per second. Since there are 5280 feet in a mile, there is good news for those still resisting the metric system, for this works out to about 982,000,000 feet per second. Thus within 2% accuracy the speed of light is 1 billion feet per second or, in more practical units, 1 foot per nanosecond. (A nanosecond (ns) is a billionth of a second.) A speed of 1 ft per nanosecond is actually relevant in setting limits on the size a computer can have if you want it to be really fast. Arithmetic operations are now being done in substantially less than a microsecond (a millionth of a second), nanosecond computers are surely just around the corner, and if you want to inform some remote part of the computer what you have just done before you do the next thing it had better not be more than a foot away, since (as we shall see) no information can be transmitted faster than the speed of light.

In discussing issues related to speeds it is very useful to use units in which the speed of light assumes an especially simple form. In 1959 the foot was officially redefined to be exactly 0.3048 of a meter. Since, the speed of light is exactly 299,792,458 meters per second, if only people in 1959 had defined the foot to be 0.299792458 of a meter, a mere 1.64% shorter, then the speed of light would now be *exactly* one foot per nanosecond (1 f/n).[9] This unit will prove to be so useful for concrete examples, that for the purposes of Physics 209 *I hereby redefine the foot:*

Henceforth by one foot we shall mean the distance light travels in a nanosecond. A foot, if you will, is a light nanosecond (and a nanosecond, even more nicely, can be viewed as a light foot.) We shall revert to the clumsier term "light nanosecond" if it ever becomes necessary to distinguish between our foot, and the conventional slightly larger foot, but I doubt that it will. If it deeply offends you to redefine the foot (as it did one referee of a paper I sent to the American Journal of Physics a few years ago) then you may define 0.299792458 meters to be one phoot, and think "phoot" whenever I say or write "foot".

There is something peculiar, and, as we shall see, extraordinary and remarkable about the unqualified assertion that the speed of light in empty space is 299,792,458 meters per second. Ordinarily when you specify a speed to such high precision and indeed when you mention any speed at all, the question "with respect to what" comes irresistibly to mind. After all the speed of an object depends on the frame of reference in which that speed is measured. A ball somebody throws while riding on a uniformly moving train has one speed with respect to the train, but quite another speed with respect to the tracks. In

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[9] Sometimes more conveniently expressed as 1000 feet per microsecond (1000 f/μ). For comparison note that the speed of sound in ordinary air is about 1000 f/s. Light travels about a million times faster than sound.

the case of light there are two obvious possible answers to the question "with respect to what?":

First obvious answer: The speed of light is 299,792,458 m/sec with respect to the source of the light. When you turn on a flashlight, the light it produces has a speed of 299,792,458 m/sec with respect to that flashlight. What else could it be? In much the same way, when one specifies the speed of a bullet, one always has in mind its speed with respect to the gun from which it has emerged.

Unfortunately this answer is contradicted by our current understanding of the electromagnetic character of light. In the 19th century there was a great unification of the laws of electricity and magnetism, completed by the Scottish physicist James Clerk Maxwell. Maxwell's equations led to the prediction that when electrically charged particles jiggled back and forth (as they do, for example inside a hot wire) they would emit radiant energy that travelled at a speed of about 300,000 kilometers per second. Since this speed was numerically indistinguishable from the speed of light, it was natural to identify light with a particular form of such radiation (associated with a very rapid jiggling — almost a million billion times a second). Maxwell's equations also unambiguously predicted that *this speed was quite independent of the speed of the source of the radiation.* The speed of the light did not depend on whether the chunk of matter in which the charged particles were jiggling was stationary or moving toward or away from the direction in which the light was emitted.

Second obvious answer: With respect to a light medium (called the ether).[10] The analogy here would be not to bullets from a gun, but to sound, which is a wave in the air. Like the speed of light, the speed of sound does not depend on the speed of the source of the sound. Sound moves at a definite speed with respect to the air, whose vibrations constitute and transmit that sound. If light is a vibration of something called the ether, then the speed of light should be with respect to that ether.

Since the Earth moves about the sun at a brisk clip of 30 km/sec in various directions, and the sun moves briskly about the center of our galaxy, it would be a remarkable coincidence if the earth just happened to be stationary in the rest frame of the ether. One would expect there to be a kind of "ether wind" blowing past the earth, leading to a dependence of the speed of light on earth on the direction of that wind.[11] Efforts to detect such a difference failed to yield a clearcut result, most famously in the the Michelson-Morley ex-

[10] I digress to remark that 299,792,458 meters per second is the speed of light *in vacuum*. Light goes significantly slower in transparent media like water or glass, and even a little bit slower in air. Therefore this ether, if it exists (it does not) must be a sort of irreducible residue of otherwise empty space—what's left after you've removed everything it is possible to remove.

[11] Thus the speed of light on earth into the direction from which the ether wind was blowing ought to be a bit less than its speed along the direction of the wind.

periment of 1887. The measurements demonstrated that if the speed of light was fixed with respect to an ether, then at the time the experiment was performed, the earth, in spite of its complicated motion with respect to the galaxy, was improbably close to being at rest in the rest frame of that ether.[12]

The importance of the Michelson-Morley experiment in the historical development of relativity has been hotly debated. In his famous 1905 paper setting forth relativity[13] Einstein alludes to it only in passing: "Examples of this sort, together with unsuccessful attempts to determine any motion of the earth relative to the 'light medium', lead to the conjecture that. . . ." The reference is hardly more than parenthetical. Such attempts have to be mentioned, because if they had unambiguously revealed a significant direction dependence to the velocity of light on earth, reflecting its motion through the ether, the theory of relativity would have been dead on arrival.

The "examples of this sort" that Einstein offers as the real motivation for his reexamination of the nature of time, are examples of the fact that the electric and magnetic behavior of particles does seem to be consistent with the principle of relativity, in spite of the widespread view that there was a preferred inertial frame of reference for electromagnetic phenomena — namely the one in which the ether was stationary. The equations of electromagnetic theory were thought by many to be valid in that frame of reference and no other. Einstein noted, in effect, that even granting that assumption, a broad range of electromagnetic phenomena seemed to play out in much the same way in frames of reference other than the preferred frame. This led him to postulate that the laws of electromagnetism were, in fact, rigorously valid in arbitrary inertial frames of reference.[14] If this postulate were valid then, Einstein noted, "the introduction of a 'luminiferous ether' will prove to be superfluous" because there would be no way of determining the rest frame of the ether by any physical experiment involving electromagnetic phenomena.

But if Maxwell's equations are valid in any inertial frame of reference, and if they predict that electromagnetic radiation and light in particular propagate at a fixed speed that is independent of the speed of the source of the light, then light must propagate at the same speed in any inertial frame of reference. The answer to the question "with respect to what?" is, as we now know, "with respect to any inertial frame you like." The speed of light in vacuum is simply 299,792,458 m/sec in any inertial frame of reference, regardless of

[12] Stubborn people considered the possibility that the earth dragged the ether in its neighborhood along with it. But if that were so then the apparent positions of stars in the sky should shift through the year depending on the way in which the ether was being dragged by the Earth. No such shift is observed.

[13] "Zur Electrodynamik Bewegter Körper" ("On the Electrodynamics of Moving Bodies"), Annalen der Physik **17**, 132-148.

[14] It is this specific postulate — that what we now call the principle of relativity applies to electromagnetism as well as to Newtonian mechanics — that Einstein named the "Principle of Relativity".

how fast the source of the light is moving, and regardless of the choice of frame of reference in which the measurement of the speed of the light is made. If, for example, you race after the light in a rocket at 10 km/sec you do not reduce its speed away from you to 299,782 km/sec. It still recedes from you at 299,792 km/sec.[15]

How can this be? How can there be a speed[16] c with the property that if something moves with speed c then it must have the speed c in any inertial frame of reference? This is highly counterintuitive. Indeed, "counterintuitive" is too weak a word. It seems downright impossible. One of the central aims of our study of relativity will be to remove this sense of impossibility, and see how it can, in fact, make perfect sense.

To do this we must look very closely and critically at what it actually means to "have a speed" with respect to a particular frame of reference. When we say that an object moves uniformly with a certain speed s, we mean that it goes a certain distance D in a certain time T and that the distance and time are related by $D/T = s$. We are therefore led, inexorably, to examine carefully how one actually measures such distances and how one actually measures such times.

Let P be a valid procedure for carrying out the time and distance measurements that allow one to determine the speed of an object in a given inertial frame. Let Bob, carrying out the procedure P in the frame of reference of a space station, measure the speed of a pulse of light as it zooms off into space. He will find that it moves at about 299,792 km/s. Suppose Alice flies swiftly after the light at a speed Bob determines to be 792 km/s. Bob will then (correctly) note that in each second the light gets an additional 299,792 km away from him and Alice gets an additional 792 km away, so that the distance between Alice and the light is growing at only 299,000 km/sec. But if Alice carries out the same procedure P in the frame of reference of her rocket ship, she will find that the speed of the light is 299,792 km/s, so that the distance between her and the light is growing at the full 299,792 km/sec.

How are we to account for this discrepancy? Obviously the methods Alice uses to measure distances and times must be different from those used by Bob. But don't they both use exactly the same procedure P? Yes, but you have to think about what "exactly the same" means. If Bob, for example, uses clocks that are stationary in the frame of his

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[15] It is only the speed of light in *vacuum* that has this special property. The speed of light in water *does* depend on how fast you are moving through the water. Indeed, what is special is not light, but the speed $c = 299{,}792{,}458$ m/s. When one simply says "speed of light" without any qualification one almost always means the speed of light in vacuum, 299,792,458 m/s.

[16] Everybody calls the speed of light in vacuum c (as in, most famously, "$E = mc^2$", about which there will be more to say later on). I always thought c stood for "constant", reflecting the fact that it doesn't vary from one frame of reference to another. But perhaps it stands for *celeritas*—Latin for speed, as in "celerity" or "accelerate".

space station to measure times, then if Alice uses *exactly* the same procedure she must use clocks that are stationary in the frame of *her* rocket ship. Thus in Bob's frame of reference Alice's clocks are moving, while his are not.[17] Similar considerations apply to the meter sticks they might use to measure distances. The not terribly subtle but easily overlooked point is that Bob's procedure *as described in Bob's frame of reference* must be exactly the same as Alice's procedure *as described in Alice's frame of reference*. But Alice's procedure *as described in Bob's frame of reference* is not exactly the same as Bob's procedure *as described in Bob's frame of reference*, and it is this difference that makes it possible for Bob to account, in an entirely rational way, for the discrepancy.[18]

The constancy of the speed of light appears paradoxical only if you assume several things, as everybody implicitly did until the year 1905, about the relation between the clocks and meter sticks used by Alice and Bob:

(1) The procedure Alice uses to synchronize all the clocks in her frame of reference gives a set of clocks that Bob agrees are synchronized when he tests them against a set of clocks that he has synchronized using the same[19] procedure in his own frame of reference.

(2) The rate of a clock, as determined in Bob's frame of reference is independent of how fast that clock moves with respect to Bob;

(3) The length of a meter stick as determined in Bob's frame of reference is independent of how fast that meter stick moves with respect to Bob,

If any of these assumptions is false, then we must reexamine the way in which the speed of an object changes as one changes the speed of the frame of reference in which the speed of that object is measured. We now know that *all three* of these assumptions are false. The special theory of relativity gives a quantitative specification of the way in which they fail, and how, when they are suitably corrected, one emerges with a simple and coherent picture of space and time measurements that is entirely in accord with the existence of an invariant speed — a speed that is the same in all inertial frames of reference.

The traditional (and simplest) way to arrive at this picture — the way we shall be taking and the way Einstein used — is simply to accept as a working hypothesis that in any inertial frame of reference, any procedure that correctly measures the speed of light in vacuum must give 299,792,458 m/s. We shall accept the strange fact that if Alice and Bob both measure the speed of the same pulse of light, they will both find it to be 299,792,458 m/s even though Alice and her measuring instruments may be moving at high speed with respect to Bob and his. By tentatively accepting this extremely peculiar fact, and insisting that the principle of relativity must nevertheless remain valid, we will be able to *deduce* the

[17] And, of course, vice-versa: in Alice's frame Bob's are moving and hers are not.
[18] And, *mutatis mutandis*, for Alice also to explain their disagreement.
[19] "Same" in that same tricky sense—that he does the same thing with respect to *his* frame of reference as Alice does with respect to *her* frame of reference.

precise way in which each of the three assumptions about the behavior of moving clocks and meter sticks must be modified. Once this is done, and the corrected versions of these three assumptions are understood, the strange fact will cease to appear strange. This will be our preoccupation for the next four or five weeks.

4. Relativistic Addition of Velocities

If Alice, a passenger on a train moving at v feet per second, can throw a ball at u feet per second, then if she throws the ball toward the front of the train, its speed w with respect to the tracks will be

$$w = u + v \tag{4.1}$$

in the same direction as the train.

This is known as the nonrelativistic velocity addition law. It is called “nonrelativistic” because it is only accurate when the speeds u and v are small compared with the speed of light. Evidently it fails to work when $u = c$ (i.e. if Alice turns on a flashlight instead of throwing a ball) for we know that the speed w of the light in the track frame will not be $c + v$ but simply c — the same value it has in the train frame!

Suppose, however, that Alice fired a gun that expelled “bullets” whose muzzle velocity u was 90% of the speed of light.[1] If the addition law (4.1) fails when $u = c$, it would be surprising if it worked well when u was $0.9c$ and in fact it does not. It turns out that both (4.1) and the frame-independence of the special velocity c are special cases of a very general rule for compounding velocities that works whether or not the speeds involved are small compared with the speed of light. This “relativistic velocity addition law” states that

$$w = \frac{u + v}{1 + \left(\frac{u}{c}\right)\left(\frac{v}{c}\right)}. \tag{4.2}$$

If u and v are both small compared with the speed of light, then u/c and v/c are both small numbers. Their product is then a small fraction of a small number, so the relativistic rule (4.2) differs from the more familiar nonrelativistic rule (4.1) only by dividing the nonrelativistic result by a number that hardly differs from 1. If, on the other hand, $u = c$, then (4.2) requires w also to be c, whatever the value of v may be.[2] Thus (4.2) is consistent with both our nonrelativistic experience[3] and the fact that the speed of light is the same in all inertial frames of reference.

We now show that the general relativistic rule (4.2) is a very direct consequence of the constancy of the velocity of light. We shall show that if the speed of light is the same in all inertial frames of reference then the addition law (4.1) must be replaced by (4.2)

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[1] The “bullets”, if you insist on getting practical about it, could be photons travelling down the train in a pipe containing a fluid in which the speed of light was only 0.9 feet per nanosecond. It is only the speed of light *in vacuum* that is the same in all frames of reference.

[2] Check this for yourself! It’s an easy algebraic exercise.

[3] I.e. our experience in dealing with situations in which all relevant speeds are small compared with the speed of light.

regardless of what kind of a moving object we are describing and regardless of how fast it is moving.[4]

To develop a strategy for deducing how velocities should be compounded, we first ask what goes wrong when we try to justify the nonrelativistic rule (4.1). The obvious way to determine the speed of an object is to determine the time it takes it to traverse a race-track of known length. Doing this requires two clocks, placed at the two ends of the race-track, to determine the exact times at which the object starts and finishes the race. To arrive at the nonrelativistic velocity addition law (4.1) we implicitly assume that people using the train frame and people using the track frame will agree on whether or not those two clocks are synchronized.[5] We also assume that they will agree on the length of the race-track between the two clocks and on the rates at which the clocks are running. The constancy of the velocity of light means that the addition law (4.1) cannot be correct for an object moving at the speed of light, and therefore it means that some of the assumptions on which (4.1) rests must be wrong. This casts doubt on the addition law for any velocities.

But if we are not allowed to make such assumptions about the basic instruments with which we measure velocities, how can we deduce the correct rule for compounding velocities? One way to arrive at the correct rule is to figure out, and then take fully into account, a set of new "relativistic" rules about clock–synchronization disagreements, rates of moving clocks, and lengths of moving measuring sticks, but this takes a bit of doing.[6]

There is a better way. We can take advantage of the fact that we know the the speed of at least one thing —light. By being clever we can use light to help us measure the speed of anything else in a way that makes no use whatever of either clocks or measuring sticks. This will then enable us to deduce the rule for how velocities change when the frame of reference changes, without assuming anything about the behavior of clocks and measuring sticks. The basic idea is simple. We let the object — call it a ball — run a race with a pulse of light — call it a photon. By comparing how far the ball goes with how far the

[4] The fact that the relation (4.2) applies even when none of the objects or frames of reference associated with u, v, or w have anything to do with light, even though the speed c appears explicitly in the denominator of (4.2), gives us an early indication that the speed c is built into the very nature of space and time. Objects that move at that special speed, move at that speed in all frames of reference, as a direct consequence of (4.2) itself. Photons in vacuuum happen to be examples of such objects.

[5] Prior to Einstein the assumption was never explicitly noted. People just took it for granted that there was nothing problematic about whether two clocks in different places were synchronized.

[6] This, in fact, is the way in which the correct relativistic addition law is usually deduced. We will eventually construct this new set of rules about clocks and measuring sticks, but as of now we don't know any of them. It nevertheless is possible to figure out the correct velocity addition law even before we have learned anything about the behavior of moving clocks and measuring sticks.

photon goes, we can figure out its speed.[7]

This neat idea runs into an immediate difficulty. Although the photon and the ball start their race in the same place they will be in different places at the end of the race. But to compare how much ground they cover we must be able to determine exactly where the ball is at the precise moment the photon reaches the finish line.[8] To do this we need two synchronized clocks, one at the finish line and one with the ball. To determine where the ball is at the moment the photon reaches the finish line, we must note where the ball is when its clock reads exactly the same time that the clock at the finish line reads at the moment the photon gets to the finish line. But we want to do the whole thing without relying on possibly unreliable clocks!

There is an easy way to avoid this problem. We simply arrange for the race not to end when the photon reaches the finish line. Instead the photon hits a mirror and bounces back the way it came, and the race ends only when the photon finally reencounters the ball, which is still moving in its original direction. Because the race now ends when the photon and the ball arrive at exactly the same place, we have disposed of the problem of where the ball is along its trajectory at the moment the photon wins the race, without having to use any clocks. The ball is precisely where the photon is.

Suppose this is all done on a train. We first describe the race using the train frame. Let the race start at the rear of the train and let the photon be reflected back toward the rear when it reaches the front. Suppose the photon meets the ball a fraction f of the way from the front of the train back to the rear.[9] The photon has gone the entire length of the train plus an additional fraction f of that length, but the ball has only gone the entire length of the train minus that same fraction f of the length. The ratio of the distance covered by the ball to the distance covered by the photon is thus $\frac{1-f}{1+f}$. But this must also be the ratio of their speeds.[10] So if we call the velocity of the ball in the train frame u, then since the speed of the photon in either direction is c,

$$\frac{u}{c} = \frac{1-f}{1+f}. \tag{4.3}$$

The people on the train have thus measured the speed of the ball without using clocks and without having to know the length of the cars[11] in their train!

[7] If, for example the photon, moving at speed c, covers ten times as much ground as the ball, then the speed of the ball must be $0.1c$.

[8] Let's take the case where the ball goes slower than the photon. Later we will see that there is something highly problematic about balls that move faster than light.

[9] For example if the train consists of 100 identical cars (numbered 1,2,3,... starting from the front) and the photon meets the ball in the passageway between cars 34 and 35, then $f = 0.34$.

[10] For example if the ball covers 1/5 the distance the photon covers, then its speed must be 1/5 the speed of the photon.

[11] They only have to be able to count cars. If the ball met the photon some fraction of

Pause to convince yourself that (4.3) really does summarize a simple and practical way to compare the velocities of two objects, which avoids using any clocks and avoids having to know any absolute distances.

It will be useful to rewrite[12] (4.3) as a relation that expresses the fraction f in terms of the velocity u of the ball and the speed of light c:

$$f = \frac{c-u}{c+u}. \tag{4.4}$$

Now we start all over again and analyze a similar race on the train, but this time using the terms of the the track frame, where the train has a velocity[13] v and the ball, a velocity w. As before, the photon and ball both start at the rear of the train, the photon reaches the front first, bounces back toward the rear, and the race ends when the photon reencounters the ball. We again want to know what fraction of the way back along the train the photon has to go before it meets the ball. We want to express this fraction entirely in terms of various speeds. This time the analysis is a bit more complicated, since the train is moving while the race goes on.

We continue to assume that the photon moves with speed c in both directions in the track frame. In a little while we are going to appeal to the constancy of the velocity of light to interpret this as *exactly* the same kind of race as the one we analyzed in the train frame.[14] Meanwhile, however, it might be a good idea to put the first race out of your mind while analyzing this one. You may think, if you want, of the photon in the second race as a new "track-frame photon" which has the speed c in the track frame, unlike the old train-frame photon, which had the speed c in the train frame. If you look at it this way (and you should for now) then there is nothing at all peculiar about the track-frame

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the way along a car, they would have to be able to compare the lengths of the two parts of the car, but they could do this without knowing the absolute length of either part by just counting up the number of times some measuring stick (of unknown length) went into both parts.

[12] Whenever I make an assertions that two expressions are equivalent (in this case the relation (3) and the relation (4)) you should always do the algebra (on a piece of paper or in your head) to convince yourself that I got it right. If the algebraic challenge proves too great, at least convince yourself that (4.3) and (4) are consistent by checking a few special cases. For example if $f = \frac{1}{2}$ then (4.3) tells us that $u/c = \frac{1}{3}$. On the other hand when $u = \frac{1}{3}c$, (4) does indeed give $f = \frac{1}{2}$.

[13] We take u, v, and w all to be positive — i.e. the ball moves to the right in the train frame, and the train and ball move to the right in the track frame — so that velocities and speeds are the same; the result we arrive at, however, turns out to be valid for positive or negative velocities.

[14] As a matter of fact we are going to interpret it as exactly the *same* race.

analysis that follows. It's just more complicated than the train-frame analysis because now the train is moving too.

To analyze the race in the track frame we shall have to talk about track-frame distances and times. We shall not, however, make any assumptions about how track-frame clocks and measuring sticks behave except that track-frame people have taken all necessary precautions to ensure that the track-frame speed of an object is indeed the track-frame distance it goes in a given track-frame time. Our goal is to end up with a relation like (4.3) or (4.4) that involves no times and lengths. The relation we seek involves only velocities, along with the fraction f of the way back along the train the photon has to go before it meets the ball.[15]

Suppose it takes a time T_0 for the photon to get from the back of the train to the mirror at the front and a time T_1 for the reflected photon to get from the front to the point a fraction f of the way back along the train where it reencounters the ball. Let L be the length of the train and let D be the distance between the front of the train and the ball at the moment the photon reaches the front of the train.[16]

Since T_0 is the time it takes the photon to get a distance D ahead of the ball and since both start in the same place, moving toward the front with speeds c and w, we must have[17]

$$D = cT_0 - wT_0. \tag{4.5}$$

On the other hand T_1 is the time it takes the photon and ball, initially a distance D apart, to get back together. Since the photon covers a distance cT_1 during this time and the ball, wT_1, we have

$$D = cT_1 + wT_1. \tag{4.6}$$

Since we don't know the value of D we shall eliminate it from these two relations. This gives us $cT_0 - wT_0 = cT_1 + wT_1$, which it is convenient to write in the form

$$\frac{T_1}{T_0} = \frac{c-w}{c+w}. \tag{4.7}$$

But of course we don't know the times T_1 and T_0 either. There is, however, a second very similar way to get at exactly the same ratio of times, by comparing what the photon

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[15] See part (3) of Figure 1 on the last page.

[16] Of course these times and distances are all unknown track-frame times and distances. But since the reasoning that follows is entirely track-frame reasoning, and since the problematic quantities D, L, T_0, and T_1 all drop out of the final result, this causes us no problems.

[17] It is important in convincing yourself of this and the assertions that follow, to keep referring to the figure on the last page. Read the caption of that figure, checking what it says against the figure itself. Only then should you start to read the argument on this page.

does, not to what the ball does, but to what the train does. Note first that T_0 is the time it takes the photon to get ahead of the rear of the train by the track-frame length of the train L. Since the photon has speed c and the train, speed v,

$$L = cT_0 - vT_0. \tag{4.8}$$

Note next that T_1 is the time it takes the photon, moving toward the rear at speed c to meet a point on the train originally a distance fL away from it that moves toward it at velocity v. Thus

$$fL = cT_1 + vT_1 \qquad (4.9).$$

We don't know the actual value of L any more than we knew the actual value of D, but we can also eliminate L from these last two equations. This gives us $cT_1 + vT_1 = f(cT_0 - vT_0)$, which gives us a second expression for the ratio of T_1 to T_0:

$$\frac{T_1}{T_0} = f\Big(\frac{c-v}{c+v}\Big). \tag{4.10}$$

Although we don't know either T_1 or T_0 this expression for their ratios must be the same as the other expression (4.7). We conclude that

$$f\Big(\frac{c-v}{c+v}\Big) = \frac{c-w}{c+w}. \tag{4.11}$$

This is the relation we need. All unknown times and distances have dropped out and we have a relation involving only the fraction f and some velocities. It follows immediately from (4.11) that the fraction f is related to the velocities v and w by

$$f = \Big(\frac{c+v}{c-v}\Big)\Big(\frac{c-w}{c+w}\Big). \tag{4.12}$$

I stress that as a piece of track-frame analysis, applicable to a race between a ball with track-frame speed w and a photon with track-frame speed c, both on a train with track-frame speed v, there is nothing at all peculiar about the analysis leading to (4.12).[18] Galileo would have been quite happy with it.[19]

[18] As a reassuring check that we haven't made some mistake in getting to (4.12), notice the following: Suppose the velocity v of the train in the track-frame were 0. Then the track frame would be the same frame as the train frame. Consequently w, the velocity of the ball in the track frame, would be the same as u, the velocity of the ball in the train frame. And indeed, if you set v to zero and take w to be u, you do get back our old train-frame result (4.4).

[19] Provided we made the train a boat.

We do something not to Galileo's liking only when we now declare that these two pieces of analysis we have now completed, are simply train-frame and track-frame analyses of *one and the same race.* In this race u is the train-frame velocity of the ball, w is the track-frame velocity of that same ball, and v is the track-frame velocity of the train. Peculiarly, however, — and this is the *only* peculiarity in the entire argument — *we are now going to insist that the track-frame speed of that one photon (in either direction) is exactly the same as the train-frame speed of that same photon (in either direction).* In both directions and in both frames that speed is c. This is an application of the principle of the constancy of the velocity of light.

But if we have been describing one and the same race in two different frames then f, the fraction of the way back from the front of the train where the photon meets the ball, must have the same value in either frame. For although there might (and indeed, as we shall see, there will) be disagreement between the two frames of reference over the length of the cars of the train, there can be no disagreement about where on the train the photon meets the ball. Their reunion could trigger an explosion, for example, that would make a smudge on the floor, which all observers in all frames could inspect later on at their leisure to confirm in which part of which car the meeting took place.

So the track-frame expression (4.12) for the fraction f must agree with the train-frame expression (4.4). Setting them equal gives us a relation between the three velocities w, u, and v:

$$\left(\frac{c+v}{c-v}\right)\left(\frac{c-w}{c+w}\right) = \frac{c-u}{c+u}. \tag{4.13}$$

It is useful to rewrite this relativistic velocity addition law in a form (like the form of (4.1), the nonrelativistic addition law) in which w appears on the left side and u and v on the right:

$$\frac{c-w}{c+w} = \left(\frac{c-u}{c+u}\right)\left(\frac{c-v}{c+v}\right). \tag{4.14}$$

This is the relativistic rule that replaces the nonrelativistic rule (4.1). Instead of *adding* u and v to get w we must *multiply* an expression involving u by an expression of the same form involving v to get a third expression of the same form involving w.

The relation between the nonrelativistic rule (4.1) and the relativistic rule (4.14) is not at all clear. To see that they are, in fact, rather simply related, one must carry out the simple algebraic exercise[20] of solving (4.14) for the velocity w of the ball in the track frame in terms of its speed u in the train frame and the speed v of the train. The result is the relativistic "addition law" stated in (4.2) above:

$$w = \frac{u+v}{1+\left(\frac{u}{c}\right)\left(\frac{v}{c}\right)}. \tag{4.15}$$

[20] Which is done for you in the Appendix at the end of this essay, in case you find it too complicated to do for yourself.

Although the two forms (4.14) and (4.15) of the velocity addition law are just different ways of expressing the same relation among the three velocities w, u, and v, it is helpful to keep them both in mind, since one form can be more useful than the other, depending on the question one is asking. Thus the form (4.15) makes immediately evident (as noted on page 2 above) why the nonrelativistic addition law $w = u + v$ becomes quite accurate when u and v are small compared with the speed of light. The form (4.14), on the other hand, reveals the following important fact:

If the speed u of the ball in the train frame and the speed v of the train in the track frame are both less than the speed of light, then both $\frac{c-u}{c+u}$ and $\frac{c-v}{c+v}$ will be numbers between 0 and 1. Since the product of two numbers between 0 and 1 is also between 0 and 1, this means that $\frac{c-w}{c+w}$ is between 0 and 1, which implies in turn that the speed w of the ball in the track frame is also less than the speed of light.[21] Thus the obvious stratagem for producing an object moving faster than light does not work: if you have a cannon that shoots balls at 90% of the speed of light, and you put it on a train moving at 90% of the speed of light, then the speed of the ball in the track frame will still be less than the speed of light. Indeed in this particular case (4.15) tells us that the speed w of the ball in the track frame will be a fraction $\frac{0.9+0.9}{1+(0.9)^2} = \frac{1.80}{1.81}$ of the speed of light — about 99.45%. This is the first indication we have found — there will be others — that no material object can travel faster than the speed of light.

For many purposes it is helpful to abstract the relativistic addition law from the context of balls, trains, and tracks, and state it in terms of the velocities of certain objects (or frames of reference) with respect to other objects (or frames of reference). Let us regard the track as an object called A, the train as an object called B, and the ball as an object called C. The velocity v of the train in the track frame we now call v_{BA} — "the velocity of B with respect to A". In the same way we call the velocity u of the ball in the train frame v_{CB}, the velocity of C with respect to B, and we call the velocity w of the ball in the track frame, v_{CA}. In this language the two forms for the addition law become

$$\frac{c - v_{CA}}{c + v_{CA}} = \Big(\frac{c - v_{CB}}{c + v_{CB}}\Big)\Big(\frac{c - v_{BA}}{c + v_{BA}}\Big). \tag{4.16}$$

and

$$v_{CA} = \frac{v_{CB} + v_{BA}}{1 + \big(\frac{v_{CB}}{c}\big)\big(\frac{v_{BA}}{c}\big)}. \tag{4.17}$$

Another advantage of (4.16) over (4.17) emerges when you consider the case in which object C is a rocket that itself emits a fourth object D. If D has speed v_{DC} with respect to C what is the speed v_{DA} of D with respect to A? In other words, what form does the addition law take when we compound three speeds instead of just two? This leads to a

[21] The conclusion that if u and v are less than c then so is w is not as immediately evident from (4.15) as it is from (4.14).

great mess if we try to answer the question using the form (4.17), but if we use the addition law in the form (4.16) we merely note the following:

The speed of D with respect to A can be arrived at by compounding the speed of D with respect to C and the speed of C with respect to A. Applying the general rule (4.14) to this case gives

$$\frac{c - v_{DA}}{c + v_{DA}} = \Big(\frac{c - v_{DC}}{c + v_{DC}}\Big)\Big(\frac{c - v_{CA}}{c + v_{CA}}\Big). \tag{4.18}$$

But now we can apply (4.14) again to express the quantity containing v_{CA} in terms of v_{CB} and v_{BA} to get

$$\frac{c - v_{DA}}{c + v_{DA}} = \Big(\frac{c - v_{DC}}{c + v_{DC}}\Big)\Big(\frac{c - v_{CB}}{c + v_{CB}}\Big)\Big(\frac{c - v_{BA}}{c + v_{BA}}\Big). \tag{4.19}$$

So to compound three speeds rather than just two, we just put a third term into the product in (4.16) to get (4.19). Evidently if D were a rocket that emitted a fifth object E, we could continue in this way, and so on indefinitely. The rule in the form (4.17) would get more and more complicated, but in the form (4.16) it would retain the same simple form.

The addition law in either of its two forms (4.17) or (4.16) continues to hold even when not all the velocities have the same sign (e.g. even when the ball moves toward the rear of the train, rather than the front). If, for example, Alice throws a ball with speed u toward the *rear* of a train that moves with positive velocity v along the track, then the velocity w of the ball along the track is given by

$$w = \frac{-u + v}{1 - \frac{u}{c}\frac{v}{c}} \tag{4.20}$$

since this is what (4.2) reduces to when u is replaced by $-u$. It is a useful exercise to check this by repeating the analysis of this essay for the case where the race starts at the front of the train rather than at the rear.

The structure of the relativistic velocity addition law in either of its two forms (4.14) or (4.15), as well as the equivalence of the two forms, is made somewhat more transparent if we agree to use units of space and time, like feet and nanoseconds, for which light in vacuum travels one spatial unit in each temporal unit. If we express all speeds in feet per nanosecond, then since c is 1 f/ns, (4.14) and (4.15) reduce to

$$\frac{1 - w}{1 + w} = \Big(\frac{1 - u}{1 + u}\Big)\Big(\frac{1 - v}{1 + v}\Big) \tag{4.21}$$

and

$$w = \frac{u + v}{1 + uv}. \tag{4.22}$$

These forms are valid provided we specify u, v, and w in feet per nanosecond or — what amounts to the same thing — provided we specify u, v, and w as fractions of the speed of light.[22]

Note that if u is 1 f/ns, then (4.22) immediately gives that w is also 1 f/ns, regardless of the value of v. Thus the principle of the constancy of the speed of light is contained in (4.22) as a special case.[23] This is hardly surprising, since we used the principle to derive (4.22) in the first place. But now we can view the principle of the constancy of the speed of light as a special case of the more general modification of the non-relativistic rule for combining velocities.

Note also that if u and v are both tiny fractions of the speed of light, then $1 + uv$ will differ from 1 by a tiny fraction of a tiny fraction, and the relativistic law (4.22) hardly differs at all from the nonrelativistic rule[24] $w = u + v$. Suppose, for example, u and v are both about one foot per millisecond (i.e. 1000 feet per second, the speed of sound in air, a quite respectable speed by ordinary standards). This is only a millionth of the speed of light (one foot per nanosecond — remember a nanosecond is a billionth of a second) so 1+uv = 1.000000000001. Therefore you can forget about the relativistic correction to $w = u + v$, since you can't possibly measure the speeds accurately enough for it to make any difference.

Here is another instructive example. If u is $\frac{2}{3}$ the speed of light and v is $\frac{3}{4}$, then (4.22) tells us that[25] w is $\frac{17}{18}$. Notice something important about this example. Although the speed u of the ball in the train frame and the speed v of the train in the station frame both exceed half the speed of light, the speed w of the ball in the station frame is still less than the speed of light (though it is rather close to it).

This has important implications for the question of how fast anything can move. In the nonrelativistic world this question has a simple answer: if any motion is possible at all, then motion is possible at arbitrarily high speeds. For suppose object A can move at speed u with respect to an identical object B; then according to the principle of relativity it must be possible for object B to move at the same speed u (and in the same direction)

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[22] It is easy to check that the expression (4.22), when substituted for w on the left side of (4.21), results in the expression on the right side of (4.21).

[23] Note that (4.21) also requires, though rather more indirectly, that w must be 1 if u is 1, regardless of the value of v. For if $u = 1$ then the right side of (4.21) must be zero. This requires the left side to be zero, which is only possible if w is also 1.

[24] Although this is obvious when the relativistic law is expressed in the form (4.22), it is not at all obvious when it is expressed in the form (4.21). This illustrates the power of having more than one way to express a fundamental relation. We shall see below that for other purposes it is the multiplicative form (4.21) which is much more transparent than the additive form (4.22).

[25] You can (and should) check that these three values also satisfy (4.21).

with respect to a third identical object C, it must be possible for C to move in the same way with respect to D, D with respect to E, and so on. Using the nonrelativistic velocity addition law, $w = u + v$, we find that if the speed of A with respect to B is u, then the speed of A with respect to C is 2u, the speed of A with respect to D is 3u, the speed of A with respect to E is 4u, etc., so that by considering enough objects, each moving at the permissible speed u with respect to the next, we can make the speed of A with respect to the last object on the list as large as we wish.

Does the relativistic velocity addition law allow us to reach the same conclusion? No, it does not.

This is not immediately evident when the relativistic law is written in the form (4.22), which leads to more and more complicated expressions as more and more objects get involved. But the form (4.21) handles this more complicated situation with ease. If just three objects, A, B, and C are involved then (4.21) applies immediately and tells us that

$$\frac{1 - v_{CA}}{1 + v_{CA}} = \Big(\frac{1 - v_{CB}}{1 + v_{CB}}\Big)\Big(\frac{1 - v_{BA}}{1 + v_{BA}}\Big). \tag{4.23}$$

If we now introduce a fourth object, D, and consider its behaviour in relation to A and C then (4.21) also immediately tells us that

$$\frac{1 - v_{DA}}{1 + v_{DA}} = \Big(\frac{1 - v_{DC}}{1 + v_{DC}}\Big)\Big(\frac{1 - v_{CA}}{1 + v_{CA}}\Big). \tag{4.24}$$

But now we can apply (4.23) to the last factor on the right of (4.24) to get

$$\frac{1 - v_{DA}}{1 + v_{DA}} = \Big(\frac{1 - v_{DC}}{1 + v_{DC}}\Big)\Big(\frac{1 - v_{CB}}{1 + v_{CB}}\Big)\Big(\frac{1 - v_{BA}}{1 + v_{BA}}\Big). \tag{4.25}$$

Evidently we can continue in this way. If we have a fifth object E then

$$\frac{1 - v_{EA}}{1 + v_{EA}} = \Big(\frac{1 - v_{ED}}{1 + v_{ED}}\Big)\Big(\frac{1 - v_{DA}}{1 + v_{DA}}\Big) \tag{4.26}$$

and therefore, combining (4.26) with (4.25),

$$\frac{1 - v_{EA}}{1 + v_{EA}} = \Big(\frac{1 - v_{ED}}{1 + v_{ED}}\Big)\Big(\frac{1 - v_{DC}}{1 + v_{DC}}\Big)\Big(\frac{1 - v_{CB}}{1 + v_{CB}}\Big)\Big(\frac{1 - v_{BA}}{1 + v_{BA}}\Big). \tag{4.27}$$

In the nonrelativistic case each new object introduces a new term in the sum, so we have:

$$v_{EA} = v_{ED} + v_{DC} + v_{CB} + v_{BA}. \tag{4.28}$$

The relativistic case has a comparable simplicity, except that the sum in (4.28) is replaced by the product in (4.27), But while you can make the left side of (4.28) as big as you desire by adding together enough (small positive) terms on the right, no matter how many factors you have in a product like (4.27), if all the velocities are positive and less than 1 f/ns, then every term in the product will be positive and non-zero. Therefore their product will be positive and non-zero, and therefore the final velocity (v_{EA} in the case of (4.27)) must be less than 1 f/ns. To be sure, if every velocity is quite close to 1 f/ns, then every term in the product on the right will be extremely small, and the term on the left will be *extremely* small, so the final velocity will have to be *extremely* close to 1 f/ns. But it can't quite get there.

Suppose, for example, you have a two-stage rocket. The compound rocket is fired 90% of the speed of light, and in the frame moving with this rocket a second rocket is fired at 90% of the speed of light. This situation is simple enough for us to use the relativistic addition law in the form (4.22). The speed of the second stage in the frame of the original launch is

$$w = \frac{0.9 + 0.9}{1 + (0.9)(0.9)} = \frac{1.8}{1.81} = 0.9945\text{f/ns}, \tag{4.29}$$

or about $99\frac{1}{2}\%$ of the speed of light.

But using (4.22) gets quite clumsy if you want to think about a 5-stage rocket, each stage of which fires the next stage at 90% of the speed of light in its own rest frame. On the other hand repeated applications of (4.21) tell us immediately that

$$\frac{1-w}{1+w} = \left(\frac{1-0.9}{1+0.9}\right)^5 = \left(\frac{0.1}{1.9}\right)^5 = \frac{1}{19^5} = .0000004. \tag{4.30}$$

This tells us that w is extremely close to 1. Therefore $1+w$ must be extremely close to 2, and we can deduce from (4.30) that $1-w$ is extremely close to .0000008, and therefore w itself is about 99.99992% of the speed of light. (Contrast this to the nonrelativistic expectation that the final stage ought to be going at 5×0.9 or about $4\frac{1}{2}$ times the speed of light.) So rockets firing rockets firing rockets firing... can't get you up to the speed of light no matter how many stages you build in.

Although we derived the relativistic velocity addition law by considering a case in which all relevant velocities were positive (the ball was moving to the right in the train frame, the train was moving to the right in the station frame, and (as a consequence) the ball was moving to the right in the station frame) it continues to hold even when some of the velocities are negative. The most convincing (and instructive) way to see this is to reanalyze the race between the ball and the photon when they start at the *front* of the train and head toward the *rear*.

There is also a more immediate way to see that the addition law make sense even when some of the velocities are negative. First note that if you have any two objects at all and X moves in a given direction with respect to Y, then Y moves with the same speed but in the opposite direction with respect to X. So for any two objects X and Y it must be that

$$v_{YX} = -v_{XY} \tag{4.31}$$

Now take the case (4.23) where all the velocities are positive,

$$\frac{1 - v_{CA}}{1 + v_{CA}} = \Big(\frac{1 - v_{CB}}{1 + v_{CB}}\Big)\Big(\frac{1 - v_{BA}}{1 + v_{BA}}\Big). \tag{4.32}$$

and We can replace the positive velocity v_{BA} in (4.32) by the negative velocity v_{AB} and then express the (4.32) in the form

$$\frac{1 - v_{CB}}{1 + v_{CB}} = \Big(\frac{1 - v_{CA}}{1 + v_{CA}}\Big)\Big(\frac{1 - v_{AB}}{1 + v_{AB}}\Big). \tag{4.33}$$

This says exactly the same thing as the original law (4.16) except that the names A and B have been switched and one of the velocities (v_{AB}) is now negative.

There is one highly non-trivial consequence of the relativistic velocity addition law that was observed in the 19th century, many decades before the special theory of relativity, and was considered to constitute an outstanding puzzle in our understanding of the behavior of light. The speed u of light in water was known to be significantly less than the speed of light in vacuum. Traditionally it is written in the form $u = c/n$ where n, called the *index of refraction* of water, is a number bigger than 1. The quantity c/n is, of course, the speed of light in *stationary* water. People were also able with great ingenuity and skill to measure the speed w of light in *moving* water. If the water moves in the same direction as the light, then the non-relativistic expectation would be that the speed w of light in water moving with speed v would just be the speed $u = c/n$ of light in stationary water (i.e. in the "water-frame") + the speed v of the water in the laboratory in which the measurement was carried out:

$$w = \frac{c}{n} + v. \tag{4.34}$$

But the experiments revealed that the speed was actually lower than this, and given to a high degree of accuracy by

$$w = \frac{c}{n} + v\Big(1 - \frac{1}{n^2}\Big). \tag{4.35}$$

This seemed to hold for various fluids, with different indices of refraction n.

That strange factor in parentheses by which v was reduced was a real puzzle, and people wracked their brains to come up with plausible ether-theoretic explanations, related to the extent to which some of the "ether" was or was not being dragged along by the moving transparent liquid.

But today we know that the answer is entirely simple. The velocity w of light in a laboratory in which the water moves with speed v is related to the velocity $u = c/n$ of the light in the frame in which the water is stationary and the velocity v of the water in the laboratory by

$$w = \frac{u+v}{1+\left(\frac{u}{c}\right)\left(\frac{v}{c}\right)}, \tag{4.36}$$

so if $u = c/n$ then

$$w = \frac{\frac{c}{n}+v}{1+\frac{1}{n}\frac{v}{c}}. \tag{4.37}$$

This appears to bear little resemblance to (4.35). To see the connection, rewrite (4.37) as an expression for the *difference* between w and $\frac{c}{n}$:

$$w - \frac{c}{n} = \frac{\frac{c}{n}+v-\frac{c}{n}\left(1+\frac{1}{n}\frac{v}{c}\right)}{1+\frac{1}{n}\frac{v}{c}} = \frac{v\left(1-\frac{1}{n^2}\right)}{1+\frac{1}{n}\frac{v}{c}}. \tag{4.38}$$

This gives us just the result $v\left(1-\frac{1}{n^2}\right)$ that (4.35) specifies for the difference between w and c/n , except that it is reduced by an additional factor $1+\frac{1}{n}\frac{v}{c}$. But the speed v of the water in the laboratory is such a tiny fraction of the speed c of light, that this factor is indistinguishable from 1. So the "mysterious" result (4.35) is, in fact, a straightforward consequence of the (extraordinary) relativistic velocity addition law.

Appendix

Write (4.14) in the form[26]

$$\frac{c-w}{c+w} = \frac{a}{b}, \tag{4.39}$$

where

$$a = (c-u)(c-v) \tag{4.40}$$

and

$$b = (c+u)(c+v). \tag{4.41}$$

It follows from (4.39) that

$$(c-w)b = (c+w)a \tag{4.42}$$

or

$$c(b-a) = w(b+a) \tag{4.43}$$

so that

$$\frac{w}{c} = \frac{b-a}{b+a}. \tag{4.44}$$

[26] The reason for doing this is simply that a and b are easier to carry through the next few steps than the more complicated expressions that they stand for.

Now according to (4.41) and (4.42)

$$\begin{aligned} b =& c^2 + c(u+v) + uv. \\ a =& c^2 - c(u+v) + uv \end{aligned} \tag{4.45}$$

and therefore

$$\begin{aligned} b + a =& 2(c^2 + uv), \\ b - a =& 2c(u+v). \end{aligned} \tag{4.46}$$

This immediately reduces (4.44) to (4.15).

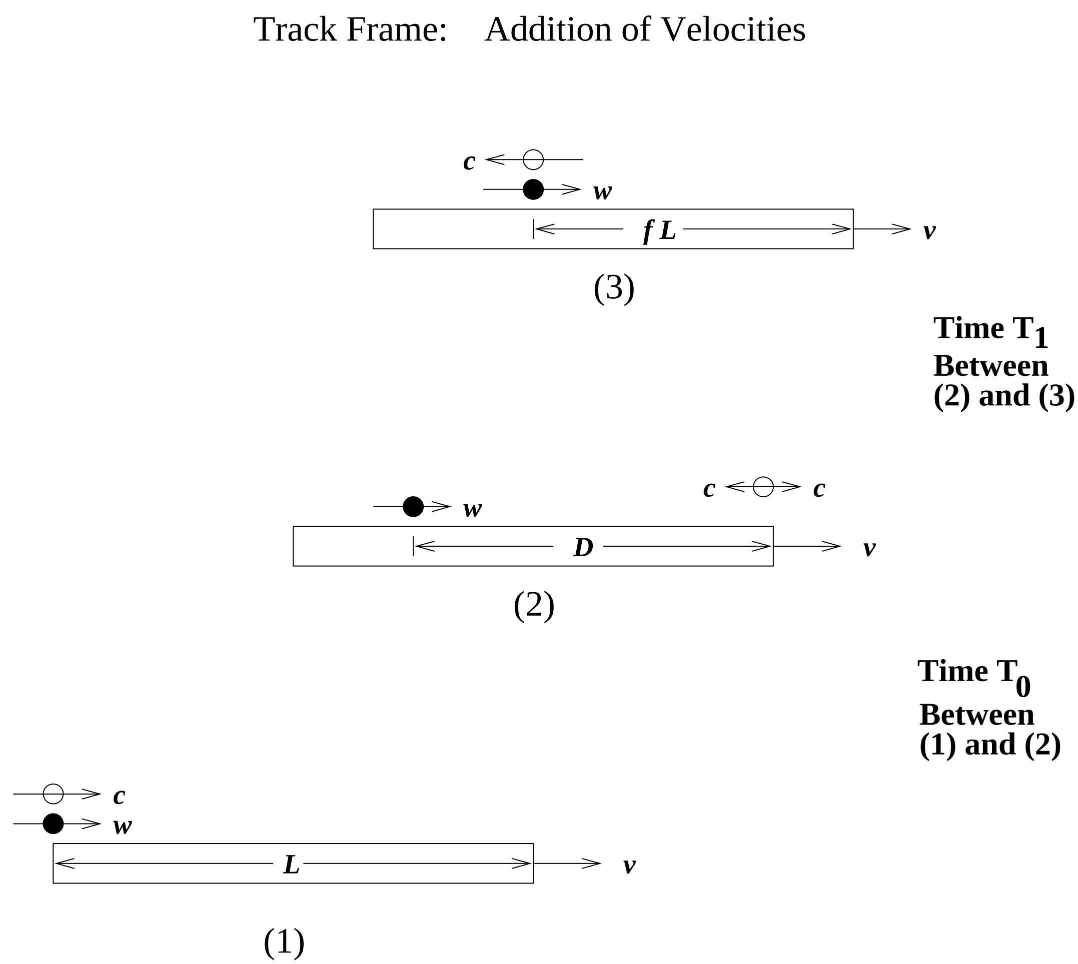

Figure 1. A photon (white circle, speed c) runs a race with a ball (black circle, speed w) in a moving train (long rectangle, speed v). The race is pictured at three different moments from the point of view of the track frame. (1) The start of the race. Photon and ball are together at the rear of the train moving with speeds c and w. The (track-frame)[27] length of the train is L (in this and the other two pictures). The train moves to the right with speed v. (2) A (track-frame) time T_0 after the events pictured in (1), the photon reaches the front of the train and bounces back toward the rear (whence the two-headed arrow). At this (track-frame) moment the photon has got a (track-frame) distance D ahead of the ball. (3) The conclusion of the race. A (track-frame) time T_1 after the events pictured in (2), the photon reencounters the ball a fraction f of the full length (track-frame) of the train — i.e. a (track-frame) distance fL — back from the front of the train.

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[27] Ignore all occurrences of "(track-frame)" in what follows. Since I said that everything is described from the point of view of the track-frame, they are unnecessary. Later it will be clearer to you why I put them in anyway.

5. Simultaneity and Clock Synchronization

Newton: *"Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external. . . ."* This is simply wrong—an old prejudice.

Two events in the train frame that happen at the same place but at different times, happen at different places in the track frame. This is clearly correct and quite banal—something we are all used to.[1]

Two events in the train frame that happen at the same time but at different places, happen at different times in the track frame. This is shocking.[2] But it is an immediate consequence of the constancy of the velocity of light, as we shall now see.[3]

The puzzlement we feel at the fact that a given pulse of light has the same speed in both the track frame and the train frame can be traced to a deeply ingrained fundamental misconception: the implicit belief that there is an absolute meaning to the simultaneity of two events happening in different places, independent of the frame of reference in which the events are described. This assumption is so pervasive in our view of the world that it is built into the very language we speak, making it extremely difficult to reexamine the question of what it actually means when we assert that two events in different places are simultaneous.

To see that two events in different places that are simultaneous in the train frame need not be simultaneous in the track frame, suppose one event consists of making a mark on the tracks (as they speed past) from the rear of the train, and the other consists of doing the same thing from the front.[4] How can people on the train persuade themselves that the two marks are made at the same time?

Well, one could provide both ends of the train with accurate clocks, and agree that each mark is made when the clock at its end of the train reads noon. But how can we be

[1] Example. Train frame: "I sat still in my seat and read the paper for 30 minutes." Track frame: "I started the paper in Boston and finished it in Providence."

[2] Read it again if you were not shocked; keep reading it until you are shocked.

[3] Note that it is obtained from the banal statement by simply interchanging the words "time" and "place". We will be encountering many other unexpected symmetries between time and space.

[4] The two events could be anything else you like. (Bells being rung at the front and rear of the train, lightning bolts striking each end, etc.) But since it turns out to be useful to mark the spot along the tracks where each event occurs, it is convenient to simplify those two events to nothing more than those two acts of marking the tracks.

sure the two clocks are properly *synchronized?* How do you know they both read noon *at the same time*.

Evidently checking the simultaneity with clocks in this way gets us nowhere, since confirming that the clocks are properly synchronized requires one to have precisely what we're trying to construct: a way of confirming that two events in different places—in this case, each clock reading noon—happen *at the same time.* This is a centrally important point. Two clocks in different places are useful only if they are properly synchronized. But "synchronized" means that the clocks have the same reading *at the same time*. Therefore you need a way to check that two events in different places are simultaneous, to check that the two clocks are synchronized. The questions of whether clocks in different places are synchronized, and of whether events in different places are simultaneous, are simply two ways of looking at the same puzzle. If you know how to answer one, you can answer the other.

Try again. One could bring the two clocks to the exact center of the train, directly confirm that they read the same when they're in the same place, and then carry them to the two ends. But how do you know that both clocks are running at the same rate as you carry them to the ends? Faced with a phenomenon as peculiar as the constancy of the velocity of light, it would be rash to assume that we knew anything about the rate at which a clock ran when it was moving.[5] The straightforward way to check on whether the clocks have done anything peculiar while being carried to the two ends of the train, would be to compare what each read when it got there with the reading of a stationary clocks at its end of the train. But we can only do this if those two stationary clocks are properly synchronized. This brings us right back to the original problem.

Ah, but suppose, even though we don't know how it might affect their rates, we moved the two clocks to the two ends in exactly the same way (except, of course, that one of the two clock-transportation procedures is executed in the opposite direction from the other.) Then however erratically its motion causes one clock to behave during the journey, the other, having experienced just the same kind of trip, will have run erratically in exactly the same way. So even if they lose or gain time because of their motion, the two clocks still agree when they arrived at the ends of the train. That method of providing both ends of the train with synchronized clocks ought to work. And it does! In the train frame.

But now we are faced with another problem. Even if we did cleverly use two such synchronized clocks to confirm that two events at the two ends of the train were simultaneous in the train frame, observers in the track frame would not agree that the the two clock-transportation procedures were identical, because in the track frame motion toward the front of the train is *not* insignificantly different from motion toward the back. For example the average speed at which each clock moves in the track frame is no longer the same (as it is in the train frame) for familiar reasons. Although people using the track frame would

[5] Later we will learn how to deal with this.

agree with somebody using the train frame that the reading of one clock, when it arrived at its end, was the same as the reading of the other, when it arrived at its own end,[6] they would have to do a rather elaborate calculation to determine whether each clock reached its end of the train *at the same time* as the other clock. That calculation would have to figure out how fast each of the clocks was moving in the track frame, and how far it had to go. It could get quite complicated. It can, however, be done and it leads to a remarkable conclusion that we now extract by a much more straightforward stratagem.

The stratagem, like our earlier stratagem for finding the new velocity addition law, avoids all possible worries about misbehaving clocks by using a method to check that two events in different places are simultaneous in the train frame that makes no use of clocks at all. This method can be easily analyzed in the track frame too. It relies only on the fact that the speed of light is always c — one foot per nanosecond — regardless of the direction the light is moving and regardless of the frame of reference in which that speed is measured.

Why, you might ask, should we build such a strange fact into our procedure for determining whether two spatially separated events are simultaneous?

If you do ask, it is only because you have forgotten why we started worrying about whether simultaneity might depend on frame of reference. It was our hope that this might lead us to a clearer understanding of the constancy of the velocity of light. So what we are doing is perfectly sensible. We *start* from the strange fact of the constancy of the velocity of light, and see what it *forces* us to conclude about the simultaneity of events. We shall find that it forces us to conclude that the simultaneity of two events in different places does indeed depend on the frame of reference in a way that can be stated simply and precisely.

Note first that people on the train can exploit the fact that light travels with a definite speed c to arrange that the two marks on the tracks are made from the two ends of the train simultaneously. They place a lamp in the middle of the train and then turn on the lamp. Light from the lamp races toward both ends of the train at the same speed c. Since the light has to travel the same distance (half the length of the train) in either direction, and moves at the same speed in either direction, it arrives at the two ends of the train *at*

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[6] This is because different frames can't disagree about things that happen *both* in the same place *and* at the same time. One could call this the principle of the invariance of coincidences. In the present case the two events that coincide are (1) a clock arriving at the rear of the train and (2) that clock indicating a particular number. A similar pair of events coincide at the front of the train. Since track observers must agree with train observers on what each clock reads at the instant it reaches its end of the train, they must agree that the clocks read the same when they reach the ends. But they do not agree that the identical readings of the clocks means that it took an identical amount of time for each clock to get to its end, since the clocks were moving at different speeds in the track frame and therefore might be running at different rates (as we shall soon see they are).

the same time. So if the making of each mark on the tracks is triggered by the arrival of the light, they will certainly be made at the same time. We have thus managed to produce a pair of simultaneous events in different places without having to make any problematic use of clocks.

But how is this procedure interpreted in the frame of the tracks? People using the track frame will certainly agree that the lamp is indeed in the center of the train, for if the train is 100 cars long and the lamp is bolted down between cars 50 and 51, then there is simply no denying that it is indeed in the center.[7] But in the track frame when the lamp is turned on and the light starts to move toward the two ends, the rear of the train moves toward the place where the light originated and the front moves away. Since the speed of the light in either direction is still c—remember we are using this strange fact, that the speed of the light is one foot per nanosecond in the track frame even though it is also one foot per nanosecond in the train frame— in the track frame it will clearly take the light less time to reach the rear of the train, which is heading toward the light to meet it, than it will take the light to reach the front of the train, which is running away as the light pursues it.

So people using the track frame will conclude that the light reaches the rear of the train before it reaches the front, and therefore that the mark in the rear is made before the mark in the front. The very same evidence that convinces people using the train frame that the marks are made simultaneously, convinces people using the track frame that they are not. *Whether or not two events in different places happen at the same time has no absolute meaning—it depends on the frame of reference in which the events are described.*[8]

Note next that people using the train frame, for whom the marks are made simultaneously, could use the arrival of the light signals to synchronize clocks at the front and rear of the train. Since people in the track-frame maintain that the mark in the rear is made *before* the mark in the front, the track people would also maintain that the synchronization procedure used by the train people had actually led to the clock in the front of the train being behind the clock in the rear.[9] *A disagreement about whether or not two events are simultaneous immediately implies a disagreement about whether or not two clocks are synchronized (and vice versa).*

It is not hard to make these disagreements quantitative. Let's analyze what has happened from the point of view of the track frame, where the train moves with speed v.

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[7] This is true even if the length of the train in the track frame is altered by its motion (as we shall soon see it is) because whatever that alteration might be, it would be exactly the same for both the front half and the rear half of the train.

[8] Notice that if you interchange time and space, that shocking assertion becomes quite humdrum: Whether or not two events at different times happen at the same place has no absolute meaning—it depends on the frame of reference in which the events are described.

[9] Make sure you understand this sentence before proceeding.

It's convenient to call the length of the train L. I emphasize that by L we mean the length of the train *in the track frame.*[10]

In Part (1) of the figure[11] the light is turned on in the middle of the train and the two pulses of light (which we shall call photons) start moving from the center toward the front and the rear.

Part (2) of the figure shows things a time T_r later, just as the rearward moving photon encounters the rear of the train, which has been moving toward it. At the instant of encounter a mark is made at the place along the tracks where the encounter takes place. During the time T_r the photon (which moves with speed c) has covered a distance cT_r. But that distance is just half the length of the train, reduced by the distance the rear of the train (which is moving toward the photon with speed v) has moved toward the photon in the time T_r. So

$$cT_r = \tfrac{1}{2}L - vT_r. \tag{5.1}$$

Part (3) of the figure shows things a (longer) time T_f after the light was turned on in Part (1). At this moment the forward moving photon encounters the front of the train, which has been moving away from it. At the instant of encounter a mark is made at the place along the tracks where the encounter takes place. During the time T_f the photon (which moves with speed c) has covered a distance cT_f. But that distance is just half the length of the train, increased by the distance the front of the train (which is moving with speed v) has moved away from the photon in the time T_f. So

$$cT_f = \tfrac{1}{2}L + vT_f. \tag{5.2}$$

We want to find the time $T = T_f - T_r$ between the making of the two marks, so it is natural to subtract the second equation from the first, since the left side then becomes $c(T_f - T_r)$ which is just cT. A second advantage of this procedure is that the unknown length L disappears from the result, which is simply

$$cT = v(T_f + T_r). \tag{5.3}$$

But what is $T_f + T_r$? Fortunately this quantity times c has a very simple meaning: $c(T_f + T_r)$ is just the sum of cT_r, the distance light travels along the track from the place

[10] Although we are used to thinking of the length of an object as being independent of the frame we measure it in, we can no longer take this for granted and, as noted earlier, we will indeed find it to be a false assumption.

[11] See the figure on page 9. It might be wise to stop at this point to examine the figure and read its caption, referring to the figure as you read, to make sure you understand both the figure and the caption.

where the lamp was turned on [shown in Part (1) of the figure] to the place on the track where it reaches the rear and the track is marked [shown in Part (2)]. And cT_f is the distance light travels in the other direction along the track from the place where the light was turned on [in Part (1)] to the place on the track where it reaches the front and the track is marked [shown in Part (3)]. Thus $c(T_f + T_r)$ is just the total distance D along the track between the two marks.

Replacing $(T_f + T_r)$ in (5.3) by D/c and dividing both sides of (5.3) by another factor of c so that T stands by itself on the left side, we have a relation between the track–frame time T between the making of the two marks and the track–frame distance D between them:

$$T = \frac{Dv}{c^2}. \tag{5.4}$$

We can abstract this into a general rule, by eliminating the talk of trains, tracks, and marks:

If two events[12] E_1 and E_2 are simultaneous in one frame of reference[13], then in a second frame of reference[14] that moves with speed v along the direction pointing from E_2 to E_1,[15] the event E_1 occurs a time Dv/c^2 earlier than the event E_2, where D is the distance between the two events in the second frame.

How big an effect is this? Suppose the two marks are 10,000 feet of track (about 2 miles) apart, and suppose the speed of the train is 100 f/s (about 70 miles per hour). Since the speed of light is a billion f/s, Dv/c^2 works out to $10{,}000 \times 100/(1{,}000{,}000{,}000)^2 =$ one trillionth of a second (one *picosecond*). The two events that are simultaneous in the train frame are a trillionth of a second apart in the track frame. Not the sort of thing you'd be likely to notice. On the other hand people who work with lasers these days are used to dealing with times a thousand times smaller than a picosecond (a *femtosecond*).

Notice something else about the general rule. I've remarked above that if you interchange time and space, the surprising fact that two frames of reference can disagree about whether two events in different places are simultaneous, turns into the commonplace fact that two frames of reference can disagree about whether two events at different times happen in the same place. If we measure time in nanoseconds and distances in feet (or use any other units in which $c = 1$) then this intriguing symmetry under the interchange of time and space becomes quantitative as well as qualitative. The rule says the following:

If two events are **simultaneous** in the train frame then in the track

[12] E_1 is the marking of the tracks from the rear of the train and E_2 is the marking of the tracks at the front.

[13] The train frame.

[14] The track frame.

[15] From the point of view of the train frame the track frame is moving backwards with speed v.

> frame the **time** between them in **nanoseconds** is equal to the **distance** between them in **feet**, multiplied by the speed v of the train along the tracks (in feet per nanosecond).

Take that statement and interchange time and space in every word[16] that appears in boldface type, making no other changes. What you get is this:

> If two events are **in the same place** in the train frame then in the track frame the **distance** between them in **feet** is equal to the **time** between them in **nanoseconds**, multiplied by the speed v of the train along the tracks (in feet per nanosecond).

The second rule is nothing more than the precise quantitative formulation of the commonplace and familiar rule for how far something with a given speed goes in a given time.

In summary:

If two flashes of light travel from the middle of a train to the two ends, then in the train frame, of course, they arrive at the two ends simultaneously. But in the track frame a flash reaches the rear a time Dv/c^2 before a flash reaches the front, where D is the distance between the two places on the track where the flashes reach the front and rear, and v is the speed of the train.[17]

A final (*important*) remark:

If the times of the two markings are recorded in the track frame by two clocks, properly synchronized in the track frame and attached to the tracks at the places where the marks are made, how do people using the train frame, for whom the two marks are made simultaneously, account for the fact that the track-frame clocks read times that differ by Dv/c^2?

Easily! They say that the reason the track–frame clocks indicate the rear mark was made a time Dv/c^2 before the forward mark is that the track–frame clock that recorded the time of the rear mark is actually *behind* the track–frame clock that recorded the time of arrival of the forward mark by just that amount: Dv/c^2. This gives us the following rule:[18]

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[16] English lacks a single word that is to "simultaneous" as space is to time ("simullocated" is what's needed), so you have to replace it by the phrase "in the same place".

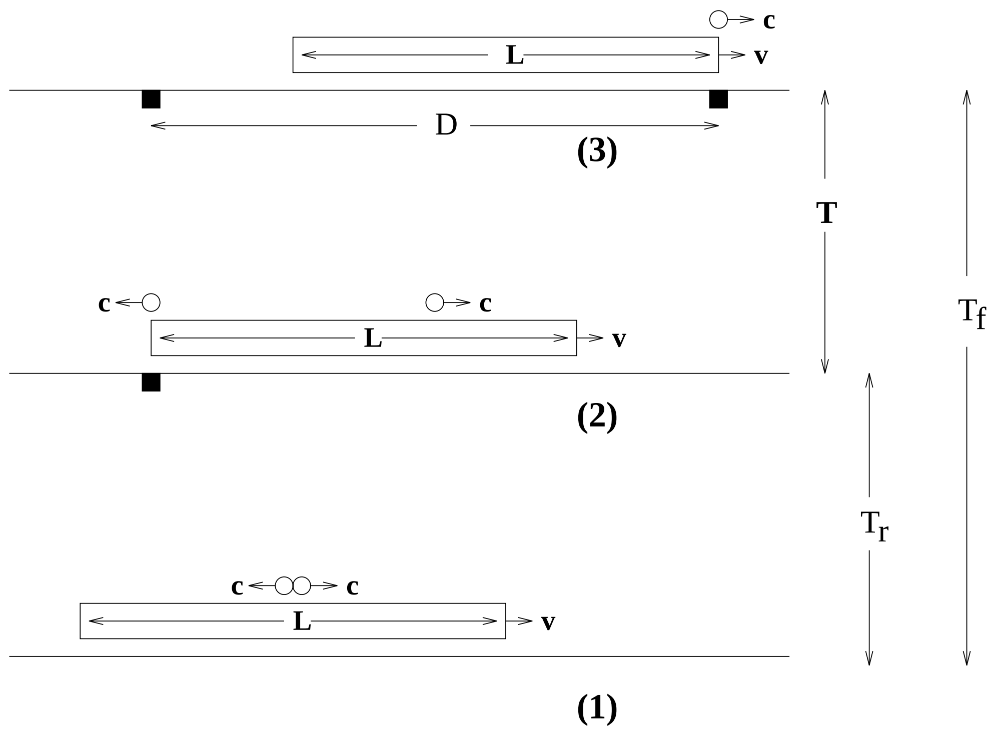
[17] It is important to emphasize that in the sentence beginning "But in the track frame. . ." the term "time" refers to track-frame time and the term "distance" refers to track-frame distance.

[18] This rule is the quantitative statement of what we had already noted qualitatively: that disagreements between frames about the simultaneity of events imply disagreements between frames about the synchronization of clocks.

If two clocks are synchronized and separated by a distance D in their proper frame,[19] *then in a frame in which the clocks move along the line joining them with speed v, the clock in front is behind the clock in the rear by Dv/c^2.*

[19] Recall that the proper frame of an object (or objects) is the frame of reference in which the object (or objects) are at rest.

The T = Dv/c^2 rule for simultaneous events



The figure depicts a series of events at three different times in the track frame. All lengths, times, and speeds shown in the figure are track-frame lengths, track-frame times, and track-frame speeds. For brevity I shall omit the phrase "track-frame" from each mention of a length or a time, but it is implicit throughout this caption.

The horizontal rectangle is a train of length L moving to the right with speed v. The white circles are photons that move with speed c. (1) Two photons originate in the center of the train, moving toward the front and the rear. (2) A time T_r after the first picture the photon on the left reaches the rear of the train. As it does so a spot (black square) is left upon the tracks to mark the place where the photon reached the rear of the train. (3) A time T_f after the first picture the photon on the right reaches the front of the train. As it does so a spot (black square) is left upon the tracks to mark the place where the photon reached the front of the train. This spot is a distance D away from the spot that was made in part (2). The time between the making of the two spots is $T = T_f - T_r$. As explained on pages 5 and 6 above, the relation between the time T between the making of the spots and the distance D between them is simply $T = Dv/c^2$.

6. Slowing down of Moving Clocks; Contraction of Moving Objects

If two clocks are synchronized and separated by a distance D in a frame in which they are at rest, then in a frame in which they move with speed v along the line joining them, they are not synchronized: the clock in front is behind the clock in the rear by an amount[1] T given by

$$T = Dv/c^2. \tag{1}$$

By exploring the consequences of this fact for two appropriately chosen clocks synchronized in the train-frame (used by Alice) and two other appropriately chosen clocks synchronized in the track-frame (used by Bob), we now deduce that moving clocks must run slowly and that moving trains (or moving tracks) must shrink along the direction of their motion. We can also deduce the precise amount by which the clocks slow down and the trains or tracks shrink.

Let the proper length of the train (i.e. the length of the train in the train (Alice's) frame) be L_A. Attached to each end of the train let there be a clock, as shown on the right half of Figure 1[2] on page 6, which depicts things as they are described in the train frame. Both clocks are synchronized in the train frame, so both read the same time: 0.

Because the clocks are synchronized in the train frame, they are not synchronized in the track frame. This is shown on the left half of Figure 1. Note that because the train clocks are *not* synchronized in the track frame, it requires *two* track frame pictures taken at two different track frame times, to depict both of them reading 0. In the upper left picture the clock at the rear of the train reads zero, and the clock in the front is behind the clock in the rear, reading a negative time $-T_A$. In the lower left picture the clock in the front of the train has advanced from $-T_A$ to 0, while the clock in the rear has advanced by the same amount[3] from 0 to $+T_A$. The track-frame time between the two pictures is

[1] A neat way to say this is that the clock in front is behind the clock in the rear by v nanoseconds per foot of separation in their proper frame, where v is the speed of the clocks in f/ns. Since v is necessarily less than 1 f/ns, this isn't an enormous effect. It's less than a microsecond per 1000 feet, and substantially less, if v is substantially less than the speed of light. If v is the speed of sound (a foot per millisecond) it's only a millionth of a microsecond per 1000 feet, or a nanosecond per million feet — half a nanosecond per thousand miles.

[2] As usual, to make sense of what I'm saying you must refer to the figure as you read the text that follows, and to read the caption of the figure before you read any further in the text.

[3] They have advanced by the same amount because they are identical clocks moving at the same speed.

the time T_B that the two clocks attached to the tracks[4] have advanced.

The quantitative rule (1) tells us that the amount T_A by which the two train-frame clocks differ in the track frame is related to the train-frame distance L_A between then by

$$T_A = L_A v/c^2. \tag{2}$$

By the same token the amount T_B by which the two track-frame clocks differ in the train frame is related to the track-frame distance D_B between them by

$$T_B = D_B v/c^2. \tag{3}$$

From this information, together with a few other simple features of Figure 1, we can deduce that moving clocks must run slowly, that moving trains or tracks must shrink along the direction of their motion, and the exact amount of the slowing-down and shrinking.

The slowing-down[5] factor for moving clocks is given by T_A/T_B. To see this look at the two track frame pictures on the left of Figure 1. Between the two pictures both track frame clocks advance by a time T_B, while both train-frame clocks advance by a time T_A. Since the track-frame clocks give correct time in the track frame, T_A is the time a train-frame clock advances in a track-frame time T_B. So the ratio T_A/T_B does indeed measure how much the moving train-frame clocks run slowly in the track frame.[6]

In the same way, the shrinking factor[7] for a moving object is given by the ratio L_B/L_A of the track-frame length L_B of the train to its length L_A (its proper length) in the frame in which it is at rest. It is also given by the ratio D_A/D_B of the train-frame length of the moving track between the two track-frame clocks and the proper length D_B of that same stretch of track:[8]

$$L_B/L_A = D_A/D_B. \tag{4}$$

To deduce the form of the shrinking and slowing-down factors we need note only two other things:

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[4] The two clocks are synchronized in the track frame, as is evident from the pictures on the left.

[5] If it turned out to be a number bigger than 1 it would be a speeding-up factor, but I'm anticipating the fact that it turns out to be less than 1.

[6] For example if the ratio T_A/T_B were 0.8 then a train-frame clock would gain only 0.8 seconds in a track-frame second.

[7] If it turned out to be a number bigger than 1 it would be a stretching factor, but I'm again anticipating the fact that it turns out to be less than 1.

[8] At the risk of being boorish, may I again remind you to check these assertions against Figure 1.

(1) It is evident from the train-frame picture on the right of Figure 1 that the train-frame length D_A of (moving) track connecting the two track frame clocks is equal to the train-frame (proper) length L_A of the train:[9]

$$L_A = D_A. \tag{5}$$

(2) The track-frame pictures on the left give a relation between L_B and D_B only slightly more complicated than (5). According to those pictures D_B is the track-frame distance between the left end of the train at track-frame time 0 and the right end at track-frame time T_B. This distance is given by the track-frame length L_B of the train *plus* the distance the train moves between the two pictures. Since the track-frame time between the two pictures is T_B and the train moves with speed v, that additional distance is vT_B, so we have

$$D_B = L_B + vT_B. \tag{6}$$

Everything we need to know follows from the relations (2)-(6). To begin with, we can conclude immediately from the relations (2), (3), and (5) that the slowing-down factor for moving clocks must be *the same* as the shrinking factor for moving sticks.[10] For (2) and (3) tell us[11] that $T_A/T_B = L_A/D_B$, while (5) tells us that $L_A = D_A$. Consequently

$$T_A/T_B = D_A/D_B. \tag{7}$$

The left side of this relation is the slowing-down factor for moving clocks; the right side is the shrinking factor for moving objects. Calling this factor s, we can write

$$T_A = sT_B, \quad D_A = sD_B, \quad \text{and also} \quad L_B = sL_A \tag{8}$$

(where the last of these follows from (4).)

To find the actual value of the shrinking (slowing-down) factor s, note that if we combine (6) with (3), we find that $D_B = L_B + v^2 D_B/c^2$, which tells us that

$$L_B = D_B(1 - v^2/c^2). \tag{9}$$

[9] Note that it is crucial in reaching this conclusion that the picture shows a single moment of train-frame time, as revealed by the fact that the train-frame synchronized clocks at the two ends of the train both read the same time 0. Were the figure, for example, a composite stitched together from fragments taken at different moments of train-frame time, then different parts of the tracks would be pictured at the places they occupied at different moments of time, and we could conclude nothing about the train-frame length of the piece of track between the clocks.

[10] Another very simple independent demonstration of this fact that makes no use of the rule (1) for synchronized clocks is given in Appendix A at the end of this essay.

[11] At the risk of being really irritating, may I remind you that when I make an assertion like this you should look back at the equations I cite to confirm that they really do tell us what I claim they do.

But (8) tells us that $L_B = sL_A$, (5) tells us that $L_A = D_A$, and (8) tells us that $D_A = sD_B$. Putting these together tells us that $L_B = s^2 D_B$, and therefore (9) tells us that

$$s^2 D_B = D_B(1 - v^2/c^2). \tag{10}$$

Consequently the shrinking factor (or slowing-down factor) is[12]

$$s = \sqrt{1 - v^2/c^2}. \tag{11}$$

This shrinking of moving objects along the direction of their motion is called the Fitzgerald contraction, in honor of the otherwise little-known Irish physicist who first suggested it. It is also called the Lorentz–Fitzgerald contraction, in honor of the great Dutch physicist H. A. Lorentz, who had the same idea at about the same time. Often it is just called the Lorentz contraction, a manifestation of the unfortunate Matthew effect.[13]

The slowing down of moving clocks is often referred to by the deplorable term "time dilation." It is deplorable because it suggests in some vague way that "time itself" (whatever that might be) is expanding. While the notion that time stretches out for a moving clock has a certain intuitive appeal, it is important to recognize that what we are actually talking about has nothing to do with any overarching concept of "time". It is simply a relation between two sets of clocks.[14] If one set of clocks is considered to be stationary, synchronized, and running at the correct rate, then a second set, considered to be moving, will be found to be both asynchronized[15] and running slowly, according to the first set. But if we consider the second set to be stationary, synchronized an running at the correct rate, then the first set will be found to be asynchronized and running slowly according to the second.

[12] Note that this is the square root of a number less than 1, so s is indeed less than 1 and is indeed a shrinking (not stretching) or slowing-down (not speeding-up) factor. Note also that if v exceeds c then (10) tells us that s^2 is a negative number, which makes no sense. Indeed, the analysis used in the lecture notes on simultaneity and clock synchronization only makes sense if the train is moving at a speed v less than the speed of light c. (In a frame in which the train moves faster than the speed of light the photon from the middle of the train will *never* reach the front of the train.) Considerations like these are further indications that the speed of light is an upper limit to how fast anything can be moving in any inertial frame of reference.

[13] "To him that hath shall be given; from him that hath not shall be taken even that which he hath."

[14] While it is commonly believed that there is something called time that is measured by clocks, I would argue that the concept of "time" is nothing more than a convenient (though potentially treacherous) device for summarizing compactly all the relationships holding between different clocks. Not all my physicist colleagues agree with me about this.

[15] Even though they are considered to be synchronized in the frame in which they are both stationary.

In both cases — the shrinking of moving sticks or the slowing down of moving clocks — one is inclined to be deeply suspicious of these conclusions. How can Alice maintain that Bob's clocks are running slowly, and Bob maintain that Alice's clocks are running slowly, when they are both talking about the same set of clocks? If Alice maintains that Bob's clocks are running slowly, shouldn't Bob necessarily maintain that Alice's clocks are running *fast*? Similarly for sticks, lined up along their direction of relative motion. If Alice maintains that Bob's moving sticks have shrunk compared with her stationary sticks, then shouldn't Bob have to maintain that Alice's moving sticks have *stretched* compared with his stationary sticks?

To succumb to this suspicion is to forget (a) that Alice and Bob also disagree on whether two events in different places happen at the same time or, equivalently, on whether two clocks in different places are synchronized. This immediately implies that each of them thinks that the other has determined the rate of a moving clock or the length of a moving stick *incorrectly*. For to measure the length of a *moving* stick one must determine where its two ends are *at the same time*[16] and this requires a judgment about whether spatially separated events at the two ends of the stick are or are not simultaneous. And to compare how fast a moving clock is running compared with stationary clocks, it is necessary to compare at least two of the readings of the moving clock with the readings of nearby stationary clocks; but since the moving clock moves, this requires one to use at least *two correctly synchronized* stationary clocks that are in two different places.

There is thus nothing inconsistent in Alice and Bob each saying that the other's clocks are running slowly and each saying that the other's sticks have shrunk, since each can point to a flaw — a failure to use properly synchronized clocks — in the procedure that the other uses to make such determinations. This is not, however, to say that the phenomena of time dilation and length contraction are mere conventions about how we use language to describe the behavior of clocks and measuring sticks. As we shall see, they can have quite striking physical consequences. It's just that the *explanations* given for those striking consequences may differ dramatically from one frame of reference to another.

One simple manifestation of this behavior, which has actually been observed, is provided by the behavior of unstable elementary particles. These have a characteristic lifetime τ. If you have a group of such particles, at rest or moving slowly, about half of them will have disintegrated within a time τ. Their collective statistical behavior therefore provides a kind of clock.

The other nice thing about such particles is that it is possible to accelerate them to speeds u very close to the speed of light.[17] When a group of such particles is travelling at

[16] If you don't get the locations of the two ends at the same time, then the stick will have moved between your two determinations of where its ends are, and you won't be getting its length right.

[17] This is done, for example, in the synchrotron at Wilson Lab on the Cornell campus.

a speed close to the speed of light c, most of the particles in the group manage to travel without disintegrating over distances much greater than the distance $u\tau$ one would expect them to be able to get if their ability to survive were unaffected by their motion. They can go much further because their "internal clocks" that govern when they decay are running much more slowly in the frame in which they rush along at speeds close to c.

Of course in the frame moving with the particles, their internal clocks are running at the normal rate, and only about half of them can survive for a time τ. However in that frame, the track along which the particles move is rushing by at close to the speed of light, distances along the track are contracted by the shrinking factor, and much more of the track can therefore go past the particles in the time τ than could have if the track were stationary.

So both frames agree that half the particles are able to cover a greater length $u\tau/s$ of track, where s is the shrinking or sllowing-down factor for things moving with the particles' velocity, which can be very small if the velocity is close to c. In the track frame this is because a typical particle survives for a time τ/s which is much longer than the time τ it would survive if it were stationary. But in the particles' frame it is because the length of the track has shrunk by a faster s so the length of moving track that can get past the particle in the time τ is augmented by the factor $1/s$. The explanatory stories differ, but the resulting behavior is the same!

This effect was observed in the behavior of μ mesons well before the age of enormous particle accelerators. These are produced by cosmic rays in the upper atmosphere. When at rest they have a lifetime of about 2 microseconds (μs), so even if they travel at the speed of light, about half of them will be gone after they have travelled 2000 feet. Yet about half the μ-mesons produced in the upper atmosphere (say 100,000 feet up) manage to make it down to the ground. This is because they travel at speeds so close to the speed of light that the slowing down factor is $s = 1/50$, and they can survive for 50 times as long as they can when stationary. In the frame of the μ-mesons, of course, their lifetime remains 2 μs, but half of them still make it down to the ground because the earth is rushing up at them so fast that the height of the atmospher contracts by a factor of $1/s = 50$, from 100,000 feet to 2,000 feet.

It is important to note that although there are substantial disagreements between Alice's train-frame picture of events and Bob's track-frame picture, whenever the two pictures are narrowed down to describe only things that happen in the same place *and* at the same time, both restricted pictures agree. This is illustrated in Figure 2. This is quite a general state of affairs. All frames of reference will agree in their description of space-time coincidences — events that happen both at the same time and in the same place. Differences of opinion only arise when it comes to "stitching" together such events to tell a more elaborate story of what things are like everywhere at a given time. The disagreements arise because "at a given time" means different things in different frames, and when this is fully taken into account, the disagreements are revealed as merely different ways of describing the same phenomena.

Finally, here is a concise summary of the basic facts about clocks and measuring sticks:

Rule for synchronized clocks: If two clocks are stationary, synchronized, and separated by a distance D in Alice's frame, then in a second frame, Bob's, in which they are moving with speed v along the line joining them, the clock in front is behind the clock in the rear by[18]

$$T = Dv/c^2. \tag{1}$$

Rule for shrinking of moving sticks or slowing down of moving clocks: The shrinking (or slowing-down) factor s associated with a speed v is given by

$$s = \sqrt{1 - v^2/c^2}. \tag{2}$$

A clock moving with speed v runs slowly by a factor s. So in T seconds according to stationary clocks, the moving clock only advances by sT seconds. Or, putting it another way, according to stationary clocks it takes t/s seconds for the reading of the moving clock to advance by t seconds.

A stick moving along its own direction with speed v shrinks by a factor s. So if the stick has proper length L, then when it moves with speed v its length is only sL.

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[18] This is equivalent to the rule for simultaneous events. For when Bob says that the clock in front is behind the clock in the rear by T, he means that the event consisting of the clock in front reading 0 (for example) is *simultaneous* with the event consisting of the clock in the rear reading T. But since the clocks are *synchronized* according to Alice, these two events, although simultaneous for Bob, are *not* simultaneous for her. Indeed, since the clocks tell correct time for Alice, in her frame the time between the two events is the difference in the clock readings: $T = Dv/c^2$. Furthermore since the clocks don't move in Alice's frame, D is the distance between two events in the history of the two clocks, even if the events are not simultaneous. Consequently Alice has an example of two events a distance D apart in her frame, that are simultaneous in Bob's frame, but are a time $T = Dv/c^2$ apart in her frame.

Appendix A

There is a simple reason why the slowing down factor s for moving clocks must be the same as the shrinking factor s for moving sticks. This explanation makes no use of the "Dv/c^2" rule for simultaneous events. Suppose (see Figure 3) we have a stick of proper length L along which a clock moves to the right with speed v. Let the clock read 0 as it passes the left end of the stick and T as it passes the right end. This is depicted in the stick-frame in the two pictures on the left of Figure 3.

The same stick and clock are depicted in the clock-frame in the two pictures on the right of Figure 3. If s is the shrinking factor for moving sticks, then in the clock frame we have a stick of length sL moving with speed v to the left. The time it takes between the left end of the stick passing the clock and the right end passing it, is just the time it takes the left end to go a distance sL to the left of the clock. Since the stick moves with speed v, that time is $T = sL/v$. Since the clock runs at its correct rate when it is stationary and reads 0 when the left end of the stick passes it, $T = sL/v$ must be the time it reads when the right end of the stick passes it.

Back in the stick frame, the clock must also read $T = sL/v$ as it passes the right end of the stick, since there can be no disagreement between frames about things that happen in the same place at the same time.[19] But in the stick frame the actual time it takes for the clock to go from the left to the right end of the stick is the length L of the stick divided by the speed v of the clock: L/v. Therefore reading of the clock has advanced by only sL/v in a stick-frame time L/v; i.e. it is running slowly by the same factor s that describes the shrinking of the moving stick in the clock frame.

[19] In this case the clock being opposite the right end of the stick and the reading on the face of the clock being $T = sL/v$.

Appendix B

There is another completely independent way to deduce the slowing down of moving clocks directly from the constancy of the velocity of light, without making any use of the Dv/c^2 rule for simultaneous events, as we have done above. I mention it here because it is one of the rare (at least in these lecture notes) examples of an argument in which *two* spatial dimensions play an important role.[20]

The argument is very simple, provided one notes two things:

(a) Suppose Alice has two synchronized clocks and Bob moves with speed v not along the direction of the line joining the two clocks, but along a direction *perpendicular* to that line. In this case Bob must agree with Alice that the clocks are synchronized. For if Alice has synchronized her clocks with light signals that originate half-way between the clocks, then because the clocks are symmetrically disposed about the direction of Bob's motion, he must agree that the two light signals reached the two clocks at the same time. This is illustrated in Figure 4. (When Bob moved *parallel* to the line joining the clocks they did not play symmetrical roles, since one moved towards and one moved away from the photon that was signalling to it. But in the present case they both move at identical angles to the trajectory of the photon that is moving towards them.)

(b) If Bob moves perpendicular to a stick of proper length L, stationary in Alice's frame, he must agree that its length is L. Moving objects do not shrink along a direction perpendicular to their directon of motion. For in contrast to the case of a measurement of length *along* the direction of motion, because of consideration (a) above, there is no ambiguity about where the two ends of the stick were *at the same time*. Bob will agree with Alice on what constitutes a valid length measurement. So if Alice concluded that Bob's meter stick was shorter than hers, then Bob would have to agree that her measurement was correct, and would therefore conclude that Alice's meter stick was longer than his. Alice would then have a rule that meter sticks moving perpendicular to their own lengths shrink, while Bob's rule would be that meter sticks moving perpendicular to their own lengths stretch. This would violate the principle of relativity.

Keeping (a) and (b) in mind, suppose Alice has a stick of proper length L, stationary in her frame, with synchronized clocks at each end of the stick. When the clocks read 0 a photon is sent from one of them to the other. Since the speed of the photon is c, it takes a time $T_A = L/c$ for the photon to travel between the clocks, so when the photon arrives at the other clock, they both read $T_A = L/c$.

But consider this from the point of view of Bob, who moves with speed v along a line perpendicular to Alice's stick. According to Bob even though the stick moves with speed

[20] And also because having another, unrelated way, to arrive at the same conclusion, bolsters one's confidence that the conclusion may actually be correct.

v, because it moves perpendicular to its length, it does not shrink and still has length L. But because the stick is moving in Bob's frame, a photon going from one end of the stick to the other has to cover a distance greater than L. Because the speed of the photon is c in Bob's frame as well as in Alice's, it must therefore takes a time T_B that is *longer* than the time $L/c = T_A$ that it takes a photon to go a distance L. Since Alice's clocks, which Bob agrees are properly synchronized, do indicate that it has taken the photon a time T_A to go from one clock to the other, Bob concludes that Alice's clocks must be running slowly.

It is easy to figure out from this what the slowing down factor must be. This is illustrated in Figure 5. During the time T_B it takes the photon to go from one clock to the other the clocks have moved a horizontal distance vT_B apart. Their vertical separation is L, and therefore the total distance $D = cT_b$ the photon has to go to get from one clock to the other must satisfy

$$D^2 = L^2 + (vT_b)^2, \tag{12}$$

by the Pythagorean theorem.

If we express D and L as cT_B and cT_A, then (12) immediately tells us that

$$T_A = T_B\sqrt{1 - v^2/c^2}\ : \tag{1}$$

the time T_A that elapses on Alice's clocks is less then the actual time, according to Bob, T_B by the slowing down factor $s = \sqrt{1 - v^2/c^2}$.

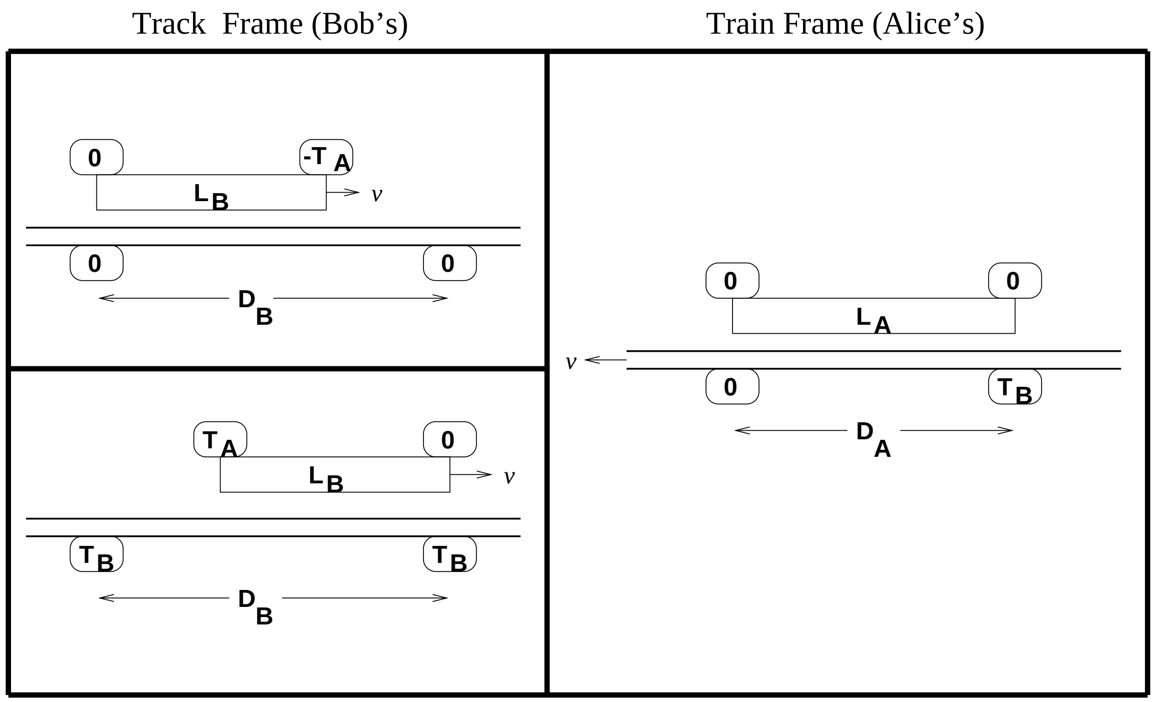

Figure 1. The figure shows three different pictures (in the three boxes bounded by heavy black lines.) Each picture shows four clocks, a train, and a track. The train is the long rectangle. Two of the clocks are attached to it, one at the front, the other at the rear (the small rounded rectangles just above the front and rear of the train.) The tracks are the two long parallel lines below the train. The other two clocks are attached to the tracks (the two small rounded rectangles shown below the tracks.) The clocks attached to the train are synchronized in the train frame; those attached to the tracks are synchronized in the track frame. The time shown by a clock is indicated by the symbol inside it.

The picture on the right depicts a single moment of time in the train frame. Both train clocks read the same time 0. The track and its attached clocks move to the left with speed v. The track clocks are not synchronized in the train frame: the clock in the front is behind the clock in the rear by a time T_B. The length of the train in the train frame is its proper length, L_A. The two clocks attached to the track are directly opposite the two clocks attached to the two ends of the train. Since the two track-frame clocks are separated by the length of the train at a single moment of train-frame time, the distance between them — the length D_A of the segment of track between them — is given by the length L_A of the train: $D_A = L_A$.

The two pictures on the left depict two different moments of time in the track frame. The first picture takes place when both track frame clocks read 0; the second takes place when both read T_B. The train and its attached clocks move to the right with speed v. Note that the train frame clocks are not syunchronized in the track frame: the clock in front is behind the clock in the rear by a time T_A. The distance between the clock attached to the tracks in the track frame is the proper length D_B of the segment of track between them. The length of the train in the track frame is L_B.

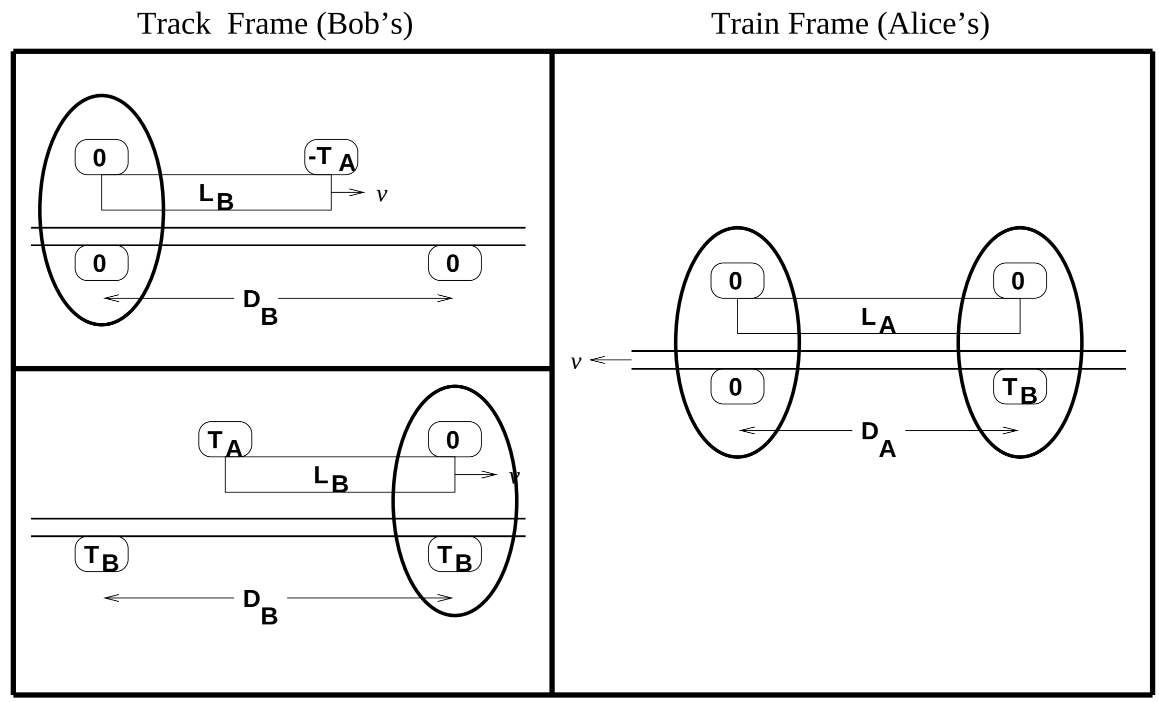

Figure 2. Figure 1 is redrawn to emphasize that although there are substantial differences of opinion between Alice and Bob (for example how long the train is, how long the stretch of track is between the two track-frame clocks, which pair of clocks is correctly synchronized, which pair of clocks is running slowly compared with which) there is complete agreement about things that happen at the same place *and* at the same time.

Thus the events that are encircled on the left of the upper track-frame picture (two clocks being together and reading 0) are described in exactly the same way in the encircled region on the left of the train-frame picture. And the two events that are encircled on the right of the lower track-frame picture (two clocks being together, the clock on the train reading 0 and the clock on the track reading T_B are also described in exactly the same way in the encircled region on the right of the train-frame picture.

There is a disagreement about whether the two events *are* (train-frame) or *are not* (track-frame) simultaneous. And there is a lot of disagreement about what is going on *somewhere else* while[21] the encircled events are happening. But there is no disagreement about the encircled events themselves.

[21] Note: "while" means "at the same time as", and is therefore a potentially treacherous word.

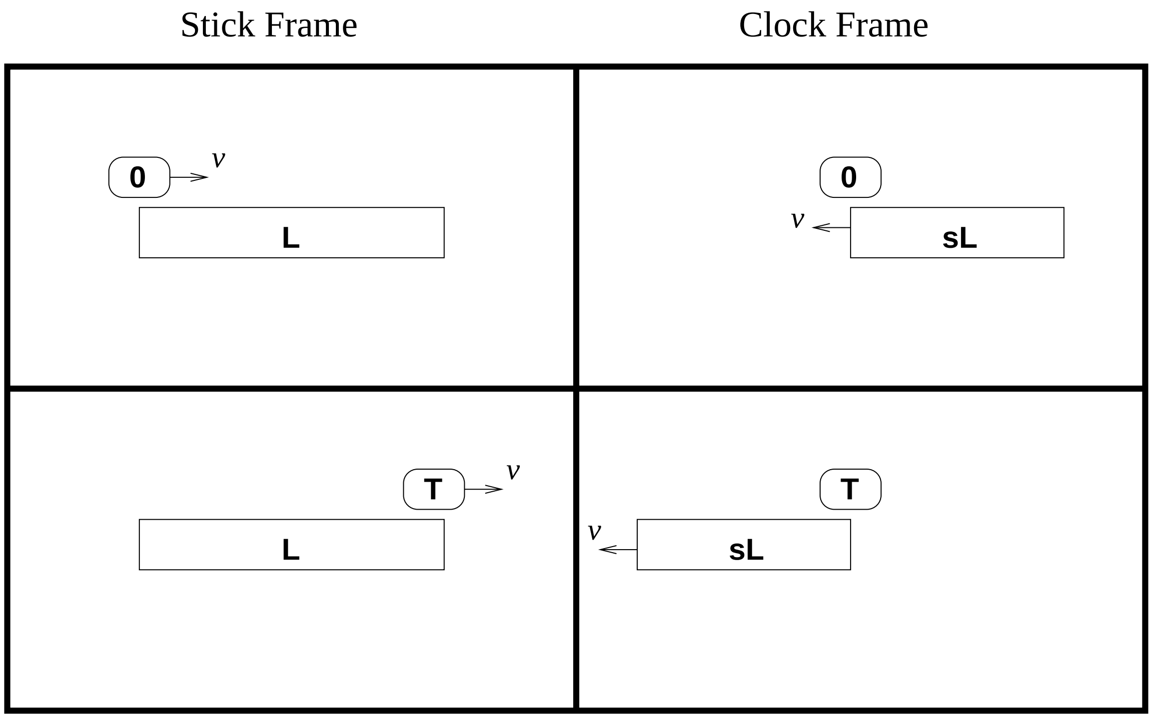

Figure 3. A stick and a clock, in relative motion. On the left we see things as described in the rest frame of the stick. The (proper) length of the stationary stick is L. The clock moves with speed v from the left end of the stick to the right end. It reads 0 when it passes the left end of the stick and T when it is passes the right end.

On the right we see things as described in the rest frame of the clock. The stick moves to the left with speed v and its length is only sL, where s is the shrinking factor. The clock is stationary. It reads 0 when the left end of the stick passes it, and T when the right end of the stick passes it. The actual value of T (in either set of pictures) is $T = sL/v$. In the clock frame this reflects the fact that the moving stick has shrunk by a factor s. In the stick frame it reflects the fact that the moving clock is running slowly by the factor s.

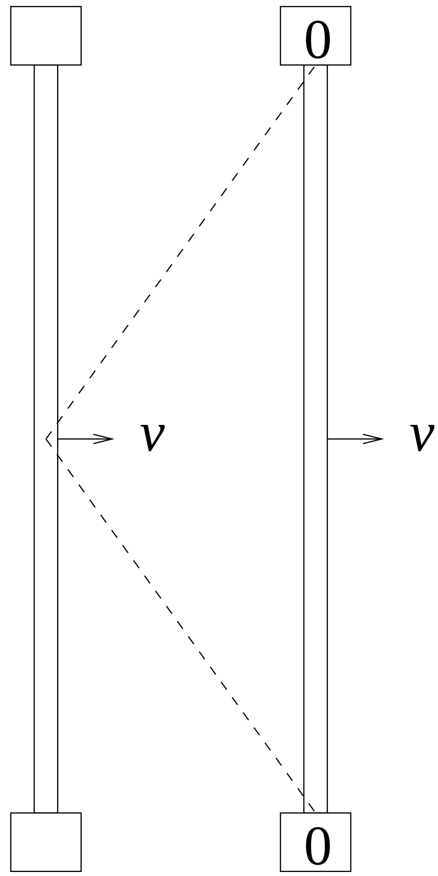

Figure 4. Alice can synchronize clocks attached to the ends of a stick by producing a pair of photons at the center of the stick that travel to the two ends. Since the photons travel the same distance at the same speed they arrive at the ends at the same time, so if Alice sets both clocks to zero when the photons arrive, they will be synchronized in her frame. The figure depicts this procedure in the frame of reference of Bob, in which the stick moves with speed v perpendicular to its length. The stick and Alice's attached clocks are shown on the left at the moment the photons are produced (the clocks have not yet been set and are shown as empty boxes), and on the right at the moment the photons arrive at the clocks and the clocks are both set to read 0. The two dashed lines are the paths traversed by the photons. It is evident that in Bob's frame, just as in Alice's, each photon has to cover the same distance to get from the center to an end of the stick, and since they both go at the same speed c, the clocks are also synchronized in his frame.

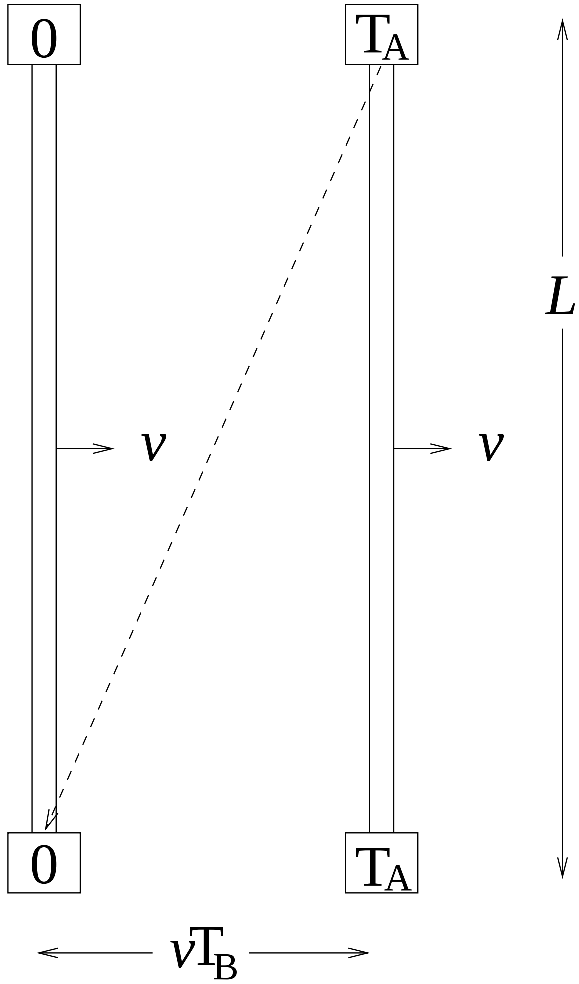

Figure 5. Alice has synchronized clocks attached to the ends of a stick of proper length L. A photon leaves one end of the stick when the clock there reads 0 and arrives at the other end when the clock there reads $T_A = L/c$. The figure shows these events as they appear in Bob's frame, in which the stick moves perpendicular to its length with speed v. The stick and Alice's attached clocks are pictured on the left, at the moment when the photon leaves one clock, reading 0, and on the right, at the moment the photon arrives at the other clock, reading T_A. Because the clocks are also synchronized in Bob's frame both of them read 0 on the left and both read T_A on the right. Because it moves perpendicular to its length the stick has the same length L in Bob's frame as it has in Alice's. If the time between the two pictures of the stick is T_B in Bob's frame, then between the pictures the stick moves a distance vT_B, and the path the photon follows from one clock to the other (dotted line) has a length $\sqrt{L^2 + (vT_B)^2}$. Since the speed of light is c in Bob's frame, this length must be cT_B, where T_B is the time between the pictures according to Bob. It follows that $T_B = (L/c)/\sqrt{1 - v^2/c^2} = T_A/\sqrt{1 - v^2/c^2}$. So according to Bob Alice's cloks are running slowly by the usual slowing-down factor.

7. Looking at a Moving Clock

We have established that in any inertial frame of reference a clock that moves with speed v runs slowly compared with stationary clocks. The slowing down factor is given by[1]

$$s = \sqrt{1 - v^2/c^2}. \tag{7.1}$$

One has the feeling that this is just some kind of trick — a conclusion based on playing sneaky intellectual games with the concept of simultaneity. If you actually *looked* at a moving clock would you actually *see* it running slowly?

The answer is that what you *see* depends on whether the clock is moving toward you or away from you. If it moves away from you, you see it running slowly; if it moves toward you, however, you see it running fast. In neither case is the rate at which you *see* it running the rate at which it actually does run (which is slowly, whether it is moving towards or away from you). The disparity between how fast it runs and how fast you see it running is a simple consequence of the fact that you do not *see* a clock reading a particular number until light that has left the clock at the moment it displays that number gets from the clock to your eyes.

If the clock is standing still this delay doesn't matter at all, because the extra time between the clock flashing each new number[2] and the light actually reaching you from each new flash is the same for each number. So there is a delay before you see each flash, but you receive the flashes at the same rate the clock is emitting them, and therefore you see the clock running at its actual rate.

But if the clock is moving away from you the light from each successive flash has further to go before it reaches you, so you see the clock running even more slowly than it actually is running. On the other hand if the clock is moving toward you, the light from each successive flash has less distance to cover, so you see the clock running faster than

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[1] Here is a possible (silly) source of confusion. Should you multiply or divide by the slowing down factor? The answer is that it depends on what the question is. The things to keep in mind are (a) that moving clocks run *slowly* and (b) that the slowing-down factor s is *less* than 1. So if the question is "How much does the reading on a moving clock advance in a time T?" the answer is that it advances by sT, since the moving clock runs slowly and multiplying T by s gives a number less than T. But if the question is how much time does it take for the reading on a moving clock to advance by T, the answer is T/s, since it takes a clock running slowly more time than T, and dividing T by a number s less than 1 produces a bigger number. Memorizing a formula doesn't work. Thinking does.

[2] It might be helpful to think of the clock as a digital clock that signals its reading in a flash of numbers. Of course even when you are looking at an ordinary mechanical clock, the only reason you can see it is that light that has bounced off its hands has then travelled (at the speed of light) to your eyes.

it actually is running It turns out (as we shall see) that this has an effect on what you actually see that is more important than the fact that the clock is running slowly. As a result when a clock moves toward you, even though it is running slowly, you *see* it running fast. And if the clock moves away from you, you *see* it running substantially more slowly than it actually is running.

It is not hard to construct a quantitative measure of this effect (known as the relativistic Doppler effect). In fact it is possible to do so without knowing the actual value (7.1) of the slowing down factor s, in a manner that establishes what that value is, quite independently of the argument we gave in Lecture Notes #6 (based on the $T = vD/c^2$ rule for simultaneous events.) The argument goes like this:[3]

Take a clock that flashes a new number every T seconds in its proper frame. Let f_tT and f_aT be the number of seconds between the flashes reaching us when the clock moves toward (t) or away (a) from us with speed v. We can deduce the values of f_t, f_a, the speeding up or slowing down factors for what we *see*, as well as the value of s, the slowing down factor for what the clock is actually doing, by the following line of thought:[4] since the moving clock runs slowly it only flashes a new number every T/s seconds. During that time it gets a distance $v(T/s)$ further away (or closer to) us, so the light from each flash takes a time $v(T/s)/c$ more (or less) to get to us. Consequently the time between light from the flashes reaching us (and therefore the time between our *seeing* successive flashes is

$$f_aT = T/s + v(T/s)/c = (T/s)(1 + v/c) \tag{7.2}$$

if the clock moves away from us and

$$f_tT = T/s - v(T/s)/c = (T/s)(1 - v/c) \tag{7.3}$$

if the clock moves toward us. Therefore

$$f_a = (1/s)(1 + v/c) \tag{7.4}$$

and

$$f_t = (1/s)(1 - v/c). \tag{7.5}$$

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[3] The argument that follows provides a derivation of the fact that $s = \sqrt{1 - v^2/c^2}$ that is independent of the analysis in Lecture Notes #6. It also provides a derivation of the relativistic velocity addition law independent of the one given in Lecture Notes #4. From a strictly logical point of view there is no need for independent arguments leading back to conclusions we have already established, but it is nevertheless reassuring to see the same conclusions emerging from quite different lines of thought.

[4] The talk that follows anticipates the fact (which we already know, but are about to rederive) that s is less than one. But the argument (suitably reworded) would work just as well if s were greater than one.

Since we already know the value of the slowing down factor s, we are finished. But even if we didn't know the value of s, we are now in a position to figure it out from the following simple but neat idea:

Suppose Alice and Bob are stationary in the same frame of reference at different places, and Bob holds a clock that Alice watches. Suppose Bob's clock flashes every second in its proper frame. Since the clock is stationary with respect to Alice, every flash takes the same time to reach her, and Alice sees a flash every second. Now suppose that Carol moves from Bob to Alice at speed v. Each time Carol sees a new number appear on Bob's clock, she reinforces it with a number of her own. She can do this automatically by setting a clock moving with her to flash a new number at the same rate that she sees Bob's numbers. Since she moves *away* from Bob with speed v she sees a flash from Bob's clock every f_a seconds. She therefore adjusts her flasher to emit a number every $T = f_a$ seconds. Since Carol and her flasher move toward Alice at speed v Alice sees Carol's flasher flashing every $f_t T = f_t f_a$ seconds. But since Carol's flashes arrive together with Bob's, and Alice sees one of Bob's flashes every second, Alice must also be seeing one of Carol's flashes every second. The net effect of Carol's seeing Bob's clock flash slowly and Alice seeing Carol's clock flash fast must therefore cancel precisely:

$$f_t f_a = 1. \tag{7.6}$$

When we combine (7.6) with (7.4) and (7.5), we learn everything of interest. Note first that (7.4) and (7.5) together require that

$$f_t f_a = (1/s)^2(1 + v/c)(1 - v/c) = (1/s)^2(1 - v^2/c^2). \tag{7.7}$$

In view of (7.6) this immediately gives us an independent confirmation of the form (7.1) of the slowing down factor s. On the other hand (7.4) and (7.5) also tell us that

$$f_t/f_a = \frac{1 - v/c}{1 + v/c}. \tag{7.8}$$

Combining this with (7.6), which tells us that $1/f_a = f_t$, we immediately learn that

$$f_t = \sqrt{\frac{1 - v/c}{1 + v/c}}, \tag{7.9}$$

and therefore f_a (which is $1/f_t$) is given by

$$f_a = \sqrt{\frac{1 + v/c}{1 - v/c}}. \tag{7.10}$$

Suppose, for example, that $v = \frac{3}{5}c$, so the slowing down factor is $\sqrt{1-(\frac{3}{5})^2} = \frac{4}{5}$. This tells us that clock moving at 60% of the speed of light takes $\frac{5}{4} = 1.25$ seconds to flash each second — it runs at $\frac{4}{5} = 80\%$ of its normal rate. But

$$\sqrt{\frac{1-\frac{3}{5}}{1+\frac{3}{5}}} = \tfrac{1}{2}, \tag{7.11}$$

so if the clock is moving toward you you see it flash a new second every half-second — i.e. you see it running at *twice* its normal rate. If it moves away from you you see it flash a new second every two seconds — i.e. you see it running at *half* its normal rate. If $v = \frac{4}{5}c$ the slowing down factor drops to $\frac{3}{5}$. But the rate at which you *see* the clock flash differs from the rate at which it runs in its proper frame by a factor of 3.

The form (7.10) for the rate at which you see a clock ticking as it moves away from you is of great importance in cosmology. Distant galaxies moving away from us contain clocks in the form of atoms — identical to those on earth — whose characteristic internal vibrations lead to the emission of light. The rate at which we see the light vibrating is slower than the rate at which the atoms vibrate in their proper frame for just the reasons we have been discussing. This slowing of the perceived vibration rate is called[5] the "red shift" (because red light vibrates more slowly than the other colors of the visible spectrum). The amount by which the vibration rate is reduced is given precisely by the relativistic result (7.10). One can therefore turn things around and use the amount of the red shift to deduce the speed at which the distant galaxy moves away from us. A red shift by a factor of 2, for example means (see Eq, (7.11)) that a galaxy is moving away at 60% of the speed of light.

Alternative derivation of the relativistic velocity addition law

By a modest generalization of the argument leading to (7.6) we can also come up with a proof of the relativistic velocity addition law entirely different from the one we constructed in Lecture Notes #4.

Suppose Bob and Charles move to the right away from Alice at speeds v and w in Alice's frame of reference, and Charles moves to the right away from Bob at speed u in Bob's frame. If Alice has a clock that flashes once a second, then Charles will receive a flash from her every $\sqrt{\frac{1+w/c}{1-w/c}}$ seconds, while Bob will receive a flash from Alice every

[5] It is an optical version of the Doppler effect, in which the pitch of a note is higher or lower depending on whether the instrument sounding the note is moving towards or away from you. Here the relativistic slowing down of the vibration in the instrument is so small as to be inconsequential, but the fact that the instrument moves toward or away from you while vibrating continues to be important. (And, of course, the speed that plays the role of c is now the speed of sound with respect to the air.)

$\sqrt{\frac{1+v/c}{1-v/c}}$ seconds. So if Bob reinforces the flashes from Alice by setting his own clock to flash at the same rate he receives Alice's flashes, then Charles will receive the reinforcing flashes from Bob every $\sqrt{\frac{1+u/c}{1-u/c}}\sqrt{\frac{1+v/c}{1-v/c}}$ seconds.[6] Since this must coincide with the rate at which he directly receives the flashes from Alice, we must have

$$\sqrt{\frac{1+w/c}{1-w/c}} = \sqrt{\frac{1+u/c}{1-u/c}}\,\sqrt{\frac{1+v/c}{1-v/c}}. \tag{7.12}$$

But (7.12) is entirely equivalent[7] to the velocity addition law written in its multiplicative form,

$$\frac{c-w}{c+w} = \Big(\frac{c-u}{c+u}\Big)\Big(\frac{c-v}{c+v}\Big). \tag{7.13}$$

[6] The factor on the left gives the number of seconds between flashes of Bob's clock in Bob's frame — namely the number of seconds between the flashes he receives from Alice. The factor on the right changes this to the number of seconds between the flashes Charles receives from Bob's flashing clock, in view of the fact that Charles is moving away from Bob with speed v.

[7] Convince yourself of this equivalence. Don't just passively take my word for it. It should be "obvious" but if you're algebraic skills are a little rusty, you might have to think about it a little.

8. Invariance of the Interval between Events

We have identified a variety of quantities that people using different inertial frames of reference disagree about: the rate of a clock, the length of a stick, whether two events are simultaneous, whether two clocks are synchronized. There are also some things people using different frames of reference do agree on: people in all frames of reference agree about whether or not two events occur both at the same time *and* the same place; people in all frames of reference agree about whether or not something moves with the speed of light c.

There is an additional class of quantities, that people using different frames of reference agree about. The constancy of the speed of light is, in fact, only a special case of this broader group of so-called invariant quantities. We can get a hint at what these invariants might be, by first giving a somewhat more abstract statement of the constancy of the speed of light:

Consider two distinct events[1] E_1 and E_2. Let D and T be the distance and time between the events in a particular frame of reference. If the two events happen to be events in the history of a single photon moving uniformly at speed c (for example the photon leaving a slide projector and arriving at a screen) then $D/T = c$.

Now since the speed of the photon is the same in all frames of reference, in any other frame of reference the distance D' and the time T' between E_1 and E_2 are also related by $D'/T' = c$, even though D' need not be equal to D nor need T' be equal to T. We can turn this into an alternative statement of the constancy of the velocity of light:

If the time T and distance D between two events are related by $D = cT$ in one frame of reference, then they will be related in the same way in any other frame of reference. Putting it another way, if the time between two events in nanoseconds is equal to the distance between them in feet in any one frame of reference, then the time between them in nanoseconds in any other frame of reference will be equal to the distance between them in feet in that other frame.

We can express this relation between the time and distance between the two events in the form[2] $(cT)^2 = D^2$ or, equivalently,

$$c^2T^2 - D^2 = 0. \tag{8.1}$$

[1] I remind you that as used in relativity, the term "event" means something that happens at a definite place and a definite time. People using different frames of reference may, of course, use different numbers to identify the place and the time of the event, but everybody will agree that the event was not something spread out over a region of space and a period of time, but something that occurred at a specific position and at a specific moment.

[2] The reason for putting this in terms of the squares is that it might sometimes be useful to define T or D to be positive or negative depending on conventions about the time order of the events or the direction from one event to the other. Since two quantities that differ only in their signs have the same squares, we can include all these alternatives by writing the relation in terms of the squares.

Two events which are separated by a time and a distance satisfying (8.1) are said to be "light-like separated" or to have a "light-like separation". The term is intended to remind you that a single photon can be present at both events—i.e. a photon can be produced at the earlier event that arrives at the later event just as the later event is taking place. Two such events can be bridged by a light signal. Using this terminology we can give an alternative statement of the constancy of the velocity of light: if two events are light-like separated in one frame of reference, they will be light-like separated in all frames of reference.

When stated in this way, the principle of the constancy of the velocity of light is a special case of a much more general principle. We show below that if T is the time and D is the distance between *any* events E_1 and E_2 in one frame of reference, then even when $c^2T^2 - D^2$ is not zero, its value is still the same in all frames of reference, even though T and D separately vary from one frame to another. This is called the (principle of the) invariance of the interval:[3]

For any pair of events a time T and a distance D apart, the value of $c^2T^2 - D^2$ does not depend on the frame of reference in which T and D are specified.

To see why this is so, we consider separately the two different ways in which $c^2T^2 - D^2$ can be non-zero: either cT is bigger than D or cT is less than D.

Suppose first that $cT > D$. Then D/T must be less than c, so it is possible for an object travelling at a speed

$$v = D/T \tag{8.2}$$

less than the speed of light, to be present at both events. The proper frame of such an object is the (unique) frame in which those two events happen in the same place. Let T_0 be the time between the two events in the frame in which they happen in the same place. One can think of T_0 as the time between the events according to a clock that is present at both of them.

If we are given the time T and distance D between the events in any frame at all, we can figure out what T_0 must be in terms of T and D from the form $s = \sqrt{1 - v^2/c^2}$ of the slowing-down factor. For in the frame in which the events are separated in space and time by D and T, a clock that is present at both events moves with the speed $v = D/T$. Therefore in the time T between the events the clock only advances by sT. So the amount T_0 the clock has advanced between the events must be related to T by

$$T_0 = sT = T\sqrt{1 - v^2/c^2}. \tag{8.3}$$

Since v is related to D and T by (8.2), it follows from (8.3) that

$$T_0^2 = T^2 - D^2/c^2. \tag{8.4}$$

[3] I comment below on the significance of the term "interval".

So when the time T and distance D between two events are related by $T > D/c$, then no matter what frame of reference you calculate it in, $T^2 - D^2/c^2$ has the same value, which is the square of the time T_0 between the two events in a special frame in which they happen at the same place.

There is also the case in which $cT < D$. Now D/T exceeds c so it is impossible for anything moving at less than the speed of light to be present at both events. There is no frame of reference in which the two events happen at the same place. Now, however, there is a frame of reference in which the two events happen at the same time!

To see why, consider two clocks that are stationary and synchronized in a frame in which the events are separated in space and time by D and T, with one clock present at each event. Since the time between the events is T in that frame and the clocks are stationary and synchronized in that frame, if the clock at the earlier event reads 0 then the clock at the later one must read T. Since the distance between the stationary clocks in that frame is D, we can arange for the clocks to be attached to the two ends of a stick of proper length D that is also stationary in that frame. In a new frame, moving with speed v along the stick in the direction from the later event to the earlier one, the clock at the later event must be behind the clock at the earlier one by Dv/c^2. If we could pick v so that Dv/c^2 were equal to T, then the events would be simultaneous in the new frame. This requires that

$$v = c^2T/D = \frac{c}{D/(cT)}. \tag{8.5}$$

Since $D > cT$ the required speed v is less than c, and there is indeed a frame in which the two events are simultaneous.

Since the two events occur at opposite ends of a stick of proper length D that is moving with speed $v = c^2T/D$ in the new frame, and since the events are simultaneous in the new frame, the distance D_0 between the two events in the new frame is just the shrunken length of the moving stick. It is therefore given by

$$D_0 = sD = D\sqrt{1 - v^2/c^2}. \tag{8.6}$$

Since the speed v of the new frame is given by (8.5) we deduce from (8.6) that

$$D_0^2 = D^2 - c^2T^2. \tag{8.7}$$

So when the time T and distance D between two events are related by $D/c > T$, then no matter what frame of reference you calculate it in, $D^2 - c^2T^2$ has the same value, which is the square of the distance D_0 between the two events in a special frame in which they happen at the same time.[4]

[4] Note the pleasing resemblance between this italicized conclusion and the italicized conclusion immediately following (8.4). When distances and times are measured in feet and nanoseconds (so that $c = 1$) the two statements differ only by the interchange of space and time.

To summarize, if D is the distance and T the time between two events then the quantity $c^2T^2 - D^2$ is independent of the frame of reference in which D and T are measured, and it is useful to distinguish between three cases:

(a) $c^2T^2 - D^2 > 0$. The events are said to be *time–like separated*, because there is a frame of reference in which they happen at the same place. In that frame they are separated *only* in time, and the time T_0 between them is given by[5] $c^2T_0^2 = c^2T^2 - D^2$.

(b) $c^2T^2 - D^2 < 0$. The events are said to be *space–like separated*, because there is a frame of reference in which they happen at the same time. In that frame they are separated *only* in position, and the distance D_0 between them is given by[6] $D_0^2 = D^2 - c^2T^2$.

(c) $c^2T^2 - D^2 = 0$. The events are said to be *light-like separated*, because a single photon can be present at both events.

The quantity

$$\sqrt{|c^2T^2 - D^2|}$$

is called the *interval* between the two events. "Interval" is a word carefully selected to be neutral as to whether the separation it suggests is in space or in time. When $c^2T^2 - D^2$ is positive, the interval between the events (divided by c)[7] is just the time between them in the frame of reference in which they happen at the same place. When $c^2T^2 - D^2$ is negative the interval between the events is just the distance between them in the frame of reference in which they happen at the same time.

There is an intriguing analogy between this state of affairs and the purely spatial description of points in a plane. Suppose we have two points P_1 and P_2 and suppose that P_1 is a distance x to the *east* of P_2, and a distance y to the *north*. Then by the Pythagorean theorem, the direct distance d between the points is given by

$$d^2 = x^2 + y^2. \tag{8.8}$$

If, on the other hand, P_1 is a distance x' to the *northeast* of P_2 and a distance y' to the *northwest*, then again by the Pythagorean theorem, the direct distance d between the

[5] Note that once you *know* that $c^2T^2 - D^2$ is independent of the frame in which D and T are measured, then it is obvious that $c^2T^2 - D^2$ is given by $c^2T_0^2$ since T_0 is the time between the events in the frame in which the distance D_0 between them is 0. It is also clear that in this case there can be no frame in which the events happen at the same time, since in such a frame T would be zero and $c^2T^2 - D^2$ could not be positive.

[6] Given that $c^2T^2 - D^2$ is indeed invariant, the value of $D^2 - c^2T^2$ is obviously D_0^2 in the frame in which the events happen at the same time, since in that frame the time T_0 between them is zero. In this case there can be no frame in which the events happen at the same place, since in such a frame D would be zero and $c^2T^2 - D^2$ could not be negative.

[7] Yet another advantage of using feet and nanoseconds as the units of space and time is that this parenthetical remark would then have been unnecessary.

points satisfies

$$d^2 = x'^2 + y'^2. \tag{8.9}$$

Since the direct distance between the points has nothing to do whether you calculate it out of eastern and northern separations or north-eastern and north-western separations, we conclude that the value of $x^2 + y^2$ doesn't depend on the purely spatial frame of reference (known as a coordinate system) used to measure x and y.

The remarkable discovery of relativity is that a similar relation holds for combined spatial and temporal separations. The only difference is that one subtracts rather than adds the squares to get the invariant quantity. The fact that an additional factor of c appears in the invariant quantity $c^2T^2 - D^2$ is not a significant difference, for if we had chosen to measure eastern separation x in one set of units (say yards) and northern separation y in another (say feet), a similar conversion factor between the units would have had to appear in the purely geometrical relation (8.8).[8] The factor c, which disappears if we use "natural units" of space and time like feet and nanoseconds, is just a conversion factor that comes in when we choose to use inappropriate units like feet and seconds (c=1,000,000,000 f/s) instead of feet and nanoseconds ($c = 1$ f/ns).

The reason nobody noticed the invariance of the interval for so long is again a consequence of the enormous size of the speed of light c on the kinds of scales we are used to using. For the kinds of temporal and spatial separations we are used to under everyday terrestrial conditions, T is simply not small enough, nor D large enough, for cT not to be enormously larger than D, so that $c^2T^2 - D^2$ is hardly distinguishable from c^2T^2. Under these circumstances the invariance of the interval reduces to the assertion that the time between any pair of events is the same in all frames of reference, which is exactly what people used to believe. Only when D becomes so large and/or T so small that D/T is no longer tiny compared with c does the invariance of the interval have the richer implications we now know it to have.

Here is an entertaining consequence of the invariance of the interval:

Consider two events in the history of a uniformly moving clock, a time T and a distance D apart. Since the distance between the two events is $D_0 = 0$ in the proper frame of the clock, the time T_0 ticked off by the clock between the events satisfies $T_0^2 = T^2 - D^2/c^2$, as we have already noted in (8.4). We can rewrite this relation in the form $T_0^2 + D^2/c^2 = T^2$ or, dividing both sides by T^2, as

$$T_0^2/T^2 + D^2/c^2T^2 = 1. \tag{8.10}$$

Since the clock is present at both events, D/T is just the speed v of the clock in the frame in which it moves; it tells us how many feet the position of the clock changes per nanosecond of time. On the other hand T_0/T tells us how many nanoseconds the clock

[8] If we continued to express d in feet, the relation would become $d^2 = 9x^2 + y^2$.

ticks off per nanosecond of time in the frame in which it moves. So (continuing to use feet and nanoseconds) we have

$$(T_0/T)^2 + v^2 = 1. \tag{8.11}$$

The relation (8.11) tells us that the sum of the square of the speed at which a uniformly moving clock runs (in nanoseconds of clock reading per nanosecond of time) plus the square of the speed at which the clock moves through space (in feet of space per nanosecond of time) is one.[9]

Now a stationary clock moves through time at one nanosecond per nanosecond and does not move through space at all. But if the clock moves, there is a tradeoff: the faster it moves through space — i.e. the larger v is — the slower it moves through time — i.e. the smaller T_0/T is — in such a way as to maintain the sum of the squares of the two at 1. It is as if the clock is always moving through a union of space and time — spacetime — at the speed of light. If the clock is stationary then the motion is entirely through time (at a speed of one nanosecond per nanosecond). But in order to move through space as well, the clock must sacrifice some of its speed through time, in order to keep the total speed through spacetime equal to 1, as required by (8.11).

The analogy with ordinary speed along a highway is striking: a car moving east with its cruise control set to a fixed speed of 55 mph must sacrifice part of its easterly speed v_e to acquire some northerly speed v_n, because the cruise control keeps the speed of the car fixed at 55 mph, while the Pythagorean theorem requires the easterly and northerly speeds to be related by $55^2 = v_e^2 + v_n^2$.

[9] You can check for yourself that this fact can also be deduced directly from the form of the slowing down factor $s = \sqrt{1 - v^2/c^2}$ for moving clocks, without exploiting the concept of interval. There is a sense in which the invariance of the interval is the deeper of the two concepts.

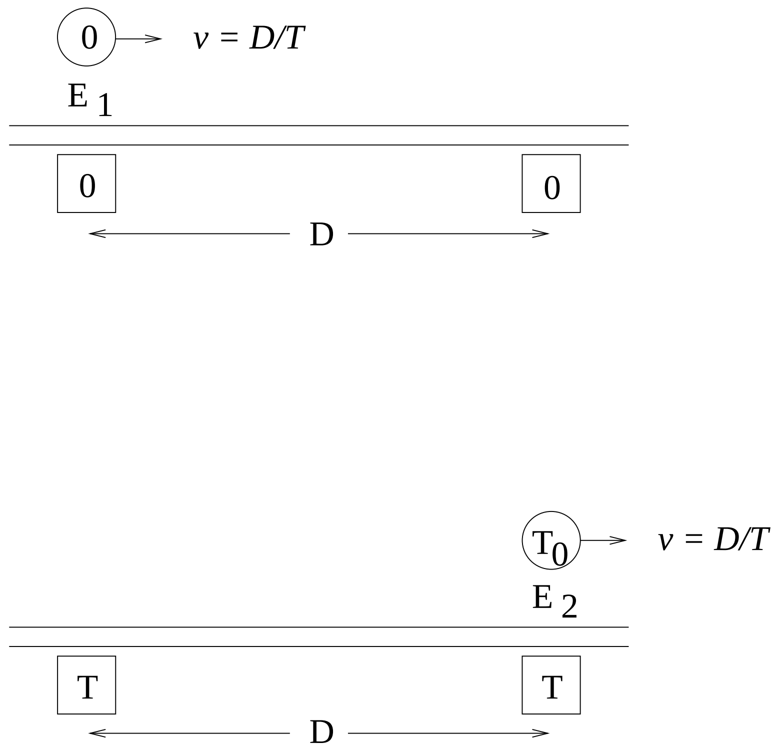


Figure 1. The invariance of the interval between time-like separated events.

The figure shows two events E_1 and E_2 that occur a time T apart and a distance D apart in the track frame. The upper part of the figure shows the track-frame situation when event E_1 occurs at track-frame time 0, as indicated by the two clocks (square boxes pictured just below the tracks) synchronized and stationary in the track frame. The lower part of the figure shows the track-frame situation when event E_2 occurs, a track-frame time T later. Both clocks have advanced by T. The track-frame distance D between the events is indicated in both parts of the figure. Because the tracks are stationary in the track frame, D is just the proper length of the portion of track stretching between the two clocks.

A third clock (the round object above the tracks) is shown, moving with speed $v = D/T$ (which is less than the speed of light c, when cT is greater than D). In the time T between the two pictures the moving clock goes a distance $vT = D$, so since it is at event E_1 in the upper picture, it has gone just the distance necessary for it to be at E_2 in the lower picture. Because the clock is moving with speed v, it runs slowly in the track frame and in a time T it only advances by $T_0 = T\sqrt{1 - v^2/c^2}$. As a result T_0 is related to T by $T_0^2 = T^2 - v^2T^2/c^2 = T^2 - D^2/c^2$.

This establishes that $T^2 - D^2/c^2$ is the time that has elapsed between the events in the special frame in which they happen at the same place.

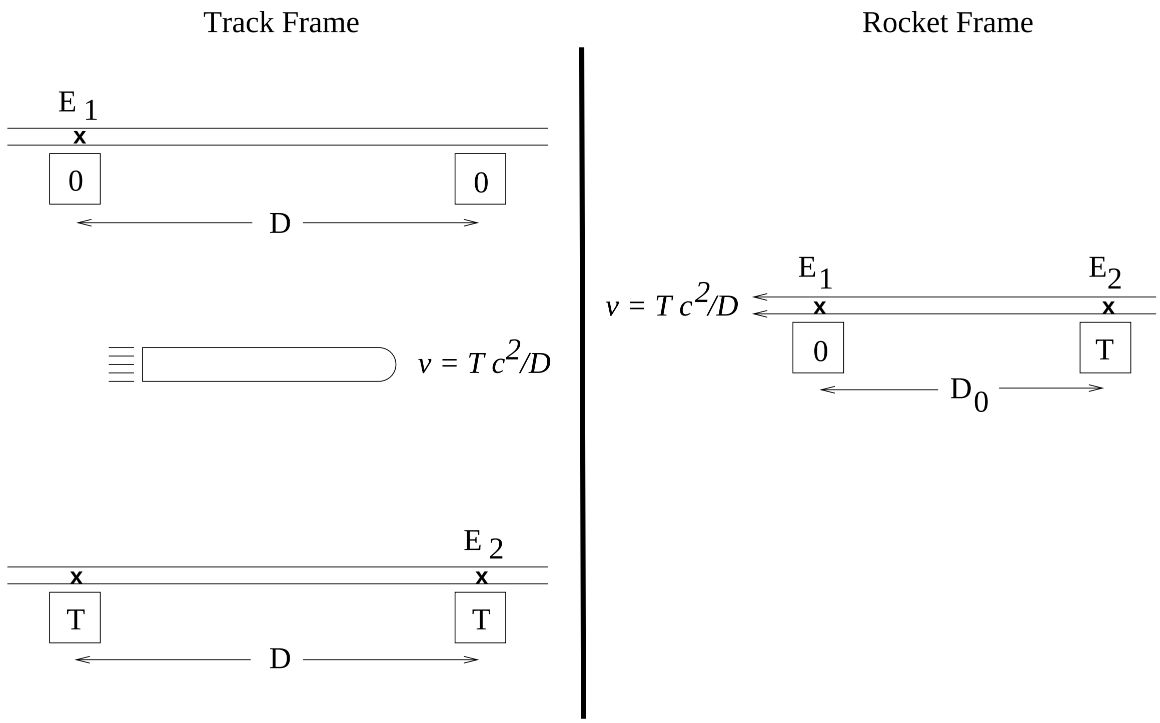


Figure 2. The invariance of the interval between space-like separated events.

The part of the figure to the left of the heavy line shows events E_1 and E_2 that occur a time T and distance D apart in the track frame. The upper part on the left shows the track-frame situation when E_1 occurs at track-frame time 0, as indicated by two clocks (square boxes pictured just below the tracks) synchronized and stationary in the track frame. The lower part on the left shows the track-frame situation when event E_2 occurs, a track-frame time T later. The track-frame distance between the events is D. Because the tracks are stationary in the track frame, D is just the proper length of the track that stretches between the two clocks. As each event occurs it makes a mark (x) on the tracks.

A rocket (the long object in the middle of the left half of the figure) is shown, moving to the right with speed $v = c(cT/D) = Tc^2/D$ (which is less than the speed of light c, when D is greater than cT). In the rocket frame the track-frame clocks move to the left with speed v and the clock on the left (the clock in front) is behind the clock on the right (the clock in the rear) by Dv/c^2. Because $v = Tc^2/D$, the clock on the left is behind the clock on the right by exactly T in the rocket frame, so in the rocket frame when the clock on the right reads T the clock on the left only reads 0. This means that the events are simultaneous in the rocket frame. Because the events are simultaneous in the rocket frame, the moving track has no time to change its position between the events, so the distance D_0 between the events is given by the distance between the two marks (x) on the moving track. This distance is the length D of track between the two marks in the track-frame, reduced by the shrinking factor $\sqrt{1 - v^2/c^2}$. As a result D_0 is related to D by $D_0^2 = D^2 - v^2D^2/c^2 = D^2 - c^2T^2$.

This establishes that $D^2 - c^2T^2$ is the distance between the events in a frame in which they happen at the same time.

9. Trains of Rockets

Here we examine a particularly simple way to see that a disagreement about whose clocks are synchronized must lead to all the relativistic effects we have been examining: the slowing down of moving clocks, the shrinking of moving sticks, the relativistic velocity addition law, the existence of an invariant velocity, and the invariance of the interval.

The trick is to examine two frames of reference from the point of view of a *third* frame in which the first two move with exactly the same speed, but in opposite directions. We take the third frame to be the proper frame of a space station, represented by the black circle in Figure 1 (shown on the last page, to make it easier to find in subsequent references to it). The other two frames are the proper frames of two trains of rockets: a grey train (moving to the left in the four parts of Figure 1 and a white train (moving to the right with the same speed).

Figure 1 shows the station and the two trains of consecutively numbered rockets at four different moments of time, as described from the point of view of the station frame. The station is in the same place in all four parts of the figure, while each train has moved an additional rocket's length from one part of the figure to the next. The three numbers preceded by a colon (e.g. :006) adjacent to each rocket represent the reading of a clock carried by that rocket. Think of each clock as being at the center of its rocket, right next to the number of the rocket.

Notice that the clocks on either train of rockets, in each of the four parts of the figure, are not synchronized: as you go towards the rear of either train the clocks get further and further ahead, the asynchronization being exactly two temporal units (which we shall call "ticks") per rocket of separation. This is in accord with the station-frame rule that if clocks have been synchronized in the train frame then they are out of synchronization in the station frame, a clock in front being behind a clock in the rear by $T = Du/c^2$ where, D is the distance between the clocks in their proper frame, and u is the speed of the train in the station frame. If we take as our unit of length the proper length of a rocket, then the figure has been drawn for a value of u such that

$$u/c^2 = 2 \text{ ticks per rocket.} \tag{9.1}$$

One can take two attitudes toward Figure 1. One can imagine that both trains are moving with such prodigious speeds and the tick is such a tiny unit of time and the clocks so very precise, that the asynchronization depicted in the Figure is the genuine relativistic effect: u/c nanoseconds of asynchronization for every foot of separation.

Alternatively, and more entertainingly, one can take the view that the rockets are moving at perfectly feasible speeds — perhaps several feet per millisecond — and the clocks are quite ordinary clocks, ticking off seconds with good but not phenomenal precision, which have been deliberately set out of synchronization by people in the space station.

The space station people wanted to test what kind of conclusions people on either train would arrive at by using unsynchronized clocks, if they failed to realize that their clocks were out of synchronization. So before the trains started to move the space station people gave the occupants of each rocket a clock, secretly setting the clocks behind by two ticks per rocket as they moved from the rear towards the front of the train, distributing the clocks. They also carefully arranged things so that people in different rockets cannot communicate with the people in other rockets of their train to compare notes on what their clocks read. The space station people have lied to the occupants of each train, falsely assuring them that clocks in different rockets are synchronized.

Once the trains are set into motion, the only information people from either train can collect, is about what is going on in their immediate vicinity. In particular when two rockets are exactly opposite each other[1] then the occupants of either rocket can note the number and clock reading of the other rocket (as well, of course, as their own). Such information can be summarized in a little figure — a photograph that people on either of the rockets might have taken — that shows just those two rockets. Figure 2, for example, shows a picture that the occupants of grey rocket 1 or white rocket 5 might have taken at the moment they were directly opposite, shown in part (d) of Figure 1. Contemplating such a picture, inhabitants of the white train would say that at a white time of 28 ticks grey rocket 1 was opposite white rocket 5 and its clock read 20 ticks. Inhabitants of the grey train, looking at the same picture, would say (equivalently) that at a grey time of 20 ticks white rocket 5 was opposite grey rocket 1 and its clock read 28 ticks. Note that the only difference in interpretation of the figures is that inhabitants of each train regard their clock as telling the correct time, and the clock on the other train as an interesting object whose reading, however, is not directly related to the time at which the picture was actually taken.

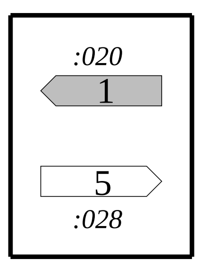


Figure 2

Suppose after the trains have gone past one another and large quantities of such information have been collected by the occupants of both trains, they return to the space station, and go off to separate rooms (a white room and a grey room) to compare notes on

[1] For example in part (a) white and grey rockets 0 are directly opposite, in part (b) grey rocket 0 is directly opposite white rocket 2, as are grey and white rockets 1, and grey rocket 2 and white rocket 0, and so on.

what pictures they took. What conclusions can they draw, acting under the assumption that the different clocks on their own train were synchronized?

The first interesting thing to examine is a pair of pictures in which the same rocket appears. Figure 3, for example, shows two pictures in which grey rocket 0 appears, taken from parts (b) and (c) of Figure 1. People on the white train will interpret these pictures as follows:

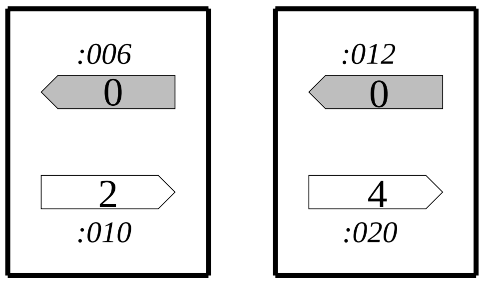


Figure 3

The most obvious thing people on the white train can read off from these two pictures is the speed of grey rocket 0, for in the first pictures it is opposite white rocket 2 at a time of 10 ticks, while in the second picture it is opposite white rocket 4 at a time of 20 ticks. So it went 2 rockets in 10 ticks, and is therefore travelling at a speed of $\frac{1}{5}$ rocket per tick. Furthermore, at the white time of 10 ticks the clock on grey rocket 0 read 6 ticks, while at the later white time of 20 ticks, the clock on grey rocket 0 read 12 ticks. Therefore in the actual white time of 10 ticks that elapsed between the taking of the two pictures, the grey clock only advanced by 6 ticks. So it is running slowly by a factor of $\frac{3}{5}$.

Note that the validity of these conclusions depends crucially on the assumption that the white clocks are synchronized, since the white people are using the readings of two *different* clocks (one in white rocket 2 and the other in white rocket 4) to make their judgments about the times at which things happened.

Since Figure 1 is completely symmetric between grey and white, the grey people will reach exactly the same conclusion about the white train and its clocks — that the train is moving at $\frac{1}{5}$ rocket per tick and its clocks are running slowly by a factor of $\frac{3}{5}$. In this way we see how a disagreement about whose clocks are correctly synchronized can lead to occupants of each of the two trains maintaining that it is the clocks on the other train that are running slowly. We, of course, taking the view of things in Figure 1 appropriate to the station frame, believe that both sets of clocks are running at the *same* rate, and that *neither* set is correctly synchronized.

Notice that from the point of view of either train, we now have both the speed v of the other train and the slowing down factor s. Anticipating that these ridiculously simple pairs of pictures extracted from the ridiculously simple set of pictures in Figure 1 are going to mimic all the relativistic effects, we can note that an s of $\frac{3}{5}$ is associated with v/c of $\frac{4}{5}$ ($s = \sqrt{1 - v^2/c^2}$). Since $v = \frac{1}{5}$ rocket per tick, we should be on the alert for the speed of

$\frac{1}{4}$ rocket per tick playing the role of an invariant velocity — the speed of light — in what follows.

The next interesting thing we can do is to examine a pair of pictures that were taken at the same time, according to one of the trains. Consider, for example, the two pictures taken at the grey time of 20 ticks, extracted from parts (c) and (d) of Figure 1, and shown in Figure 4. Because these pictures were taken at the same time, according to the occupants of the grey train, they immediately reveal that the clocks on the white train are not synchronized. At the grey time of 20 ticks the clock in white rocket 0 read 12 ticks, but that in white rocket 5 read 28 ticks. The white clocks disagree by 16 ticks and are 5 white rockets apart, so they are out of synchronization by $\frac{16}{5} = 3.2$ ticks per rocket.[2]

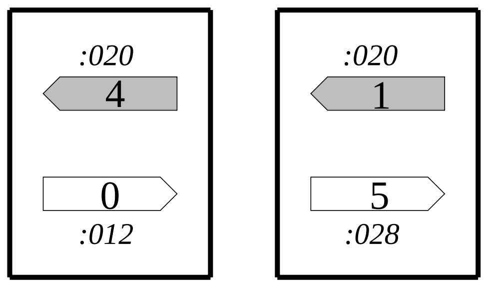


Figure 4

Furthermore people on the grey train can conclude that at a single moment of grey time — 20 ticks — five white rockets (rockets 4,3,2,1, and half each of rockets 5 and 0) stretched the same length as three grey rockets (rockets 2, 3, and half each of rockets 4 and 1) so the white rockets have shrunk by the same factor of $\frac{3}{5}$ as the white clocks are running slowly.

Notice that this amount of clock asynchronization is precisely what one would expect from the rule $T = Dv/c^2$ with $v = \frac{1}{5}$ rocket per tick and $c = \frac{1}{4}$ rocket per tick, for v/c^2 is then

$$\frac{\frac{1}{5}}{(\frac{1}{4})^2} = \frac{16}{5} = 3.2 \text{ ticks per rocket}, \tag{9.2}$$

exactly as we read off directly from Figure 4.

On the other hand, according to people using the station frame the clock asynchronization on *both* trains is 2 ticks per rocket. If the invariant velocity is indeed $c = \frac{1}{4}$ rocket per tick, and if u is the speed of either train in the station frame, then u/c^2 should be 2 ticks per rocket, which means that u, the speed of either train in the station frame, ought

[2] This is different from the asynchronization of exactly 2 ticks per rocket evident in the station frame (Figure 1), but that is to be expected, since people using the station frame know that the grey clocks are as badly out of synchronization as the white ones, and therefore the conclusions reached by the occupants of the grey train in such matters are unreliable.

to be $\frac{1}{8}$ rocket per tick. But the speed of a train in the station frame is the same as the speed of the station in the train frame, and it is evident from parts (a) and (b) of Figure 1 that the speed of the station in the frame of either train is indeed $\frac{1}{8}$ rocket per tick, since the station is opposite rocket 0 at a time of 0 ticks, and opposite tocket 1 at a time of 8 ticks (in either the white or the grey train's frame).

We can check that these various speeds are consistent with the relativistic velocity addition law,

$$v_{wg} = \frac{v_{ws} + v_{sg}}{1 + (v_{ws}v_{sg}/c^2)}, \tag{9.3}$$

where v_{wg} is the velocity of the white train in the frame of the grey train, v_{ws} is the velocity of the white train in the station frame, and v_{sg} is the velocity of the station in the frame of the grey train. We have $v_{ws} = v_{sg} = \frac{1}{8}$ rocket per tick and $c = \frac{1}{4}$ rocket per tick. When these numbers are put into (9.3) the result is indeed, $v_{wg} = \frac{1}{5}$ rocket per tick, so all the relativistic relations continue to hold.

I pause to emphasize again how very little has gone into the construction of Figure 1. The structure of part (a) is extremely simple. The only peculiar thing about it is the fact that the clocks do not all agree with each other, but the manner in which they disagree is extremely simple.evident. And the rule for getting each of the other parts from the part above it is simply to shift each train by one rocket in the direction it is going in, and advance every clock on each train by 6 ticks. Nothing elaborate has to be done to get relativity out of the figures. Once one introduces the asynchronized clocks on each train, everything else follows automatically.

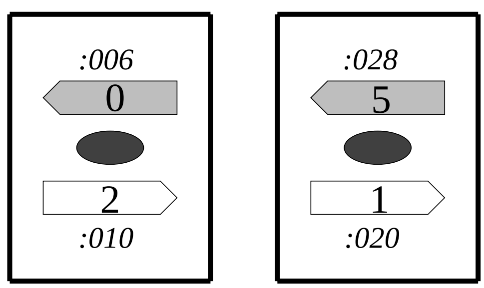


Figure 5

The relativistic velocity addition law, for example, works for anything that moves between the two trains — not just the station itself. Consider, for example, an object that was between grey rocket 0 and white rocket 2 in part (b) of Figure 1 and between grey rocket 5 and white rocket 1 in part (d). It has been captured in the two pictures shown in Figure 5. According to the grey train the object has gone 5 rockets to the right in 22 ticks, and according to the white train it has gone 1 rocket to the right in 10 ticks, so we have $v_{og} = \frac{5}{22}$ rocket per tick and $v_{ow} = \frac{1}{10}$ rocket per tick. We should have

$$v_{wg} = \frac{v_{wo} + v_{og}}{1 + (v_{wo}v_{og}/c^2)}, \tag{9.4}$$

which gives

$$v_{wg} = \frac{-\frac{1}{10} + \frac{5}{22}}{1 - (\frac{1}{10})(\frac{5}{22})/(\frac{1}{4})^2}, \tag{9.5}$$

which does indeed give $v_{wg} = \frac{1}{5}$ rocket per tick after all the arithmetic is carried out.

You can (and should) check for yourself that any other pair of pictures extracted from Figure 1 containing two moments in the history of a single object, yields values of v_{wo} and v_{og} that are consistent with the relativistic velocity addition law (9.4) and the facts that $v_{wg} = \frac{1}{5}$ rocket per tick and $c = \frac{1}{4}$ rocket per tick.

In particular, it is instructive to hunt around for a pair of photographs displaying two moments in the history of a single object moving at the special speed of $\frac{1}{4}$ rocket per tick. Figure 6 shows such a pair, taken from parts (c) and (d) of Figure 1. According to the grey train the object has moved 3 rockets to the right in a time of 12 ticks, so its velocity is $\frac{1}{4}$ rocket per tick. And according to the white train it has moved 1 rocket to the right in a time of 4 ticks, so its velocity is again $\frac{1}{4}$ rocket per tick. Such an object has the amusing ability to exploit the differences in clock synchronization on the two trains, in such a way that it can move along either train at the same speed, $\frac{1}{4}$ rocket per tick, provided the speed along a given train is timed by using the clocks carried by the rockets in that train, and provided those clocks are assumed to be synchronized.

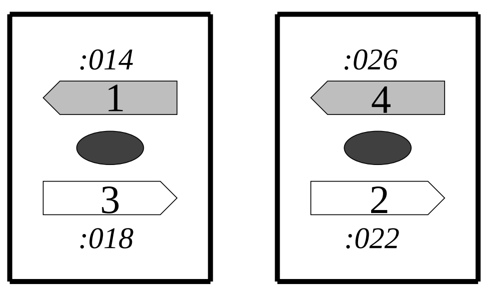


Figure 6

Figure 1 also provides us with an entirely different insight into why motion faster than light is highly problematic. Figure 7 shows two pictures in the history of a hypothetical faster-than-light object taken from parts (c) and (d) of Figure 1. According to the grey train it has gone 6 rockets in 18 ticks, for a speed of $\frac{1}{3}$ rocket per tick, which exceeds the invariant velocity $c = \frac{1}{4}$ rocket per tick. People from the white train agree that the object goes faster than the invariant velocity, having gone 4 rockets in a mere 2 ticks, for a speed of 2 rockets per tick.[3]

There is a disturbing feature to Figure 7: according to the grey train the picture on the left was taken 18 ticks before the one on the right. But according to the white train,

[3] You can check that even these superluminal velocities are consistent with the relativistic velocity addition law — but you have to be careful with the signs that indicate which way the object is moving in the frame of each train.

the picture on the left was taken 2 ticks *after* the one on the right. Occupants of the two trains disagree about the order in which the two pictures were taken! This is the kind of disagreement it is hard to tolerate. Suppose, for example, that the object were a burning candle. Its pictures would then clearly reveal the direction of time: the later the picture, the shorter the candle and the bigger the puddle of wax beneath it. Such a pair of pictures would clearly reveal to one of the groups that the clocks on its own train could not be telling the correct time.

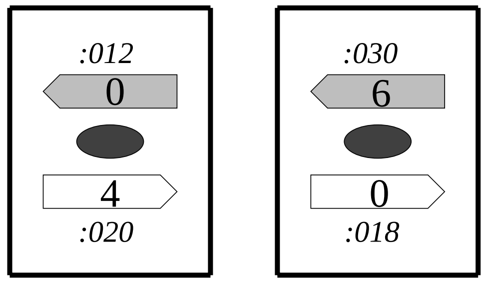


Figure 7

It turns out that this situation is quite general. If an object moves faster than light then there are always two frames of reference that disagree about the order in time of any pair of events in the history of the object.[4] Therefore if anything could move faster than light[5] it would have to be a sort of featureless blob, incapable of revealing, through its internal structure, any information about the direction of the flow of time. Burning candles, melting ice-cubes, rotting bananas, running-down batteries, aging people, and the like, cannot move faster than light.

Note, finally, that Figure 1 can also be used to demonstrate the invariance of the interval between two events. Take any pair of pictures whatever, and calculate

$$T^2 - D^2/c^2 = T^2 - (4D)^2 \tag{9.6}$$

(the 4 comes from the c^2, since c is $\frac{1}{4}$ of a rocket per tick), where T is the number of ticks between the events and D, the number of rockets. The answers will not depend on which frame you take to evaluate T and D. Consider, for example, Figure 8, which takes one event from part (b) and another from part (d) of Figure 1. According to the grey frame the two events are 22 ticks and 5 rockets apart, and $22^2-(4\times5)^2 = 22^2-20^2 = 84$. According to the white frame the two events are 10 ticks and 1 rocket apart, and $10^2-(4\times1)^2 = 10^2-4^2 = 84$. This particular pair of events is time-like separated, since T^2-D^2/c^2 is positive, and indeed, an object present at both events would have a speed less than $\frac{1}{4}$ rocket per tick in either

[4] This is most easily demonstrated using the space-time diagrams that we will soon be developing.

[5] We have already seen from our application of the velocity addition law to rockets firing rockets firing rockets that the obvious way to bring this about does not work.

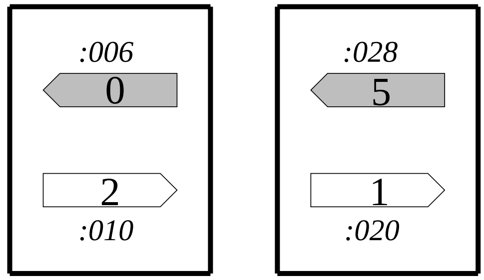


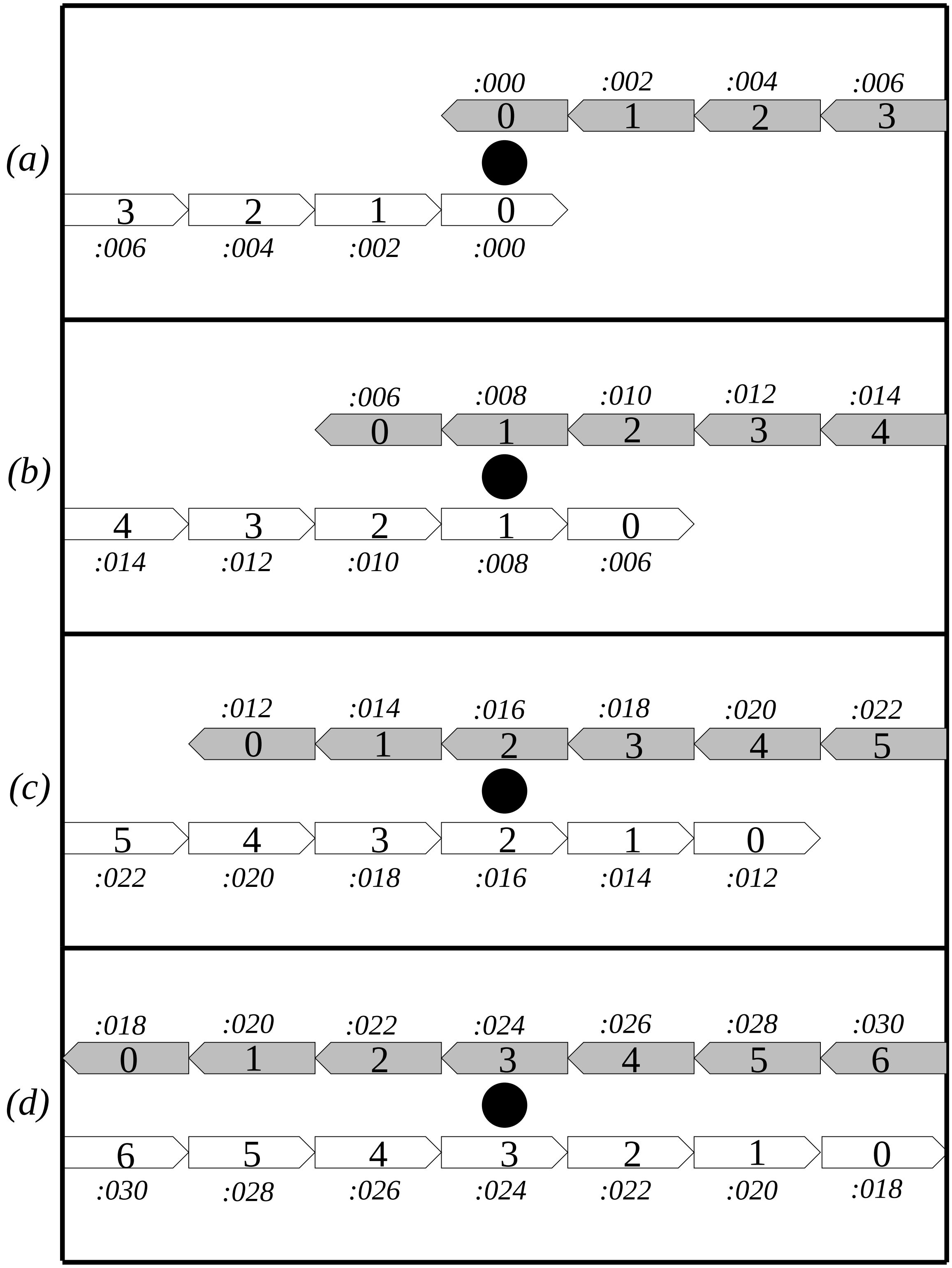
Figure 8

frame ($\frac{5}{22}$ rocket per tick in the grey frame and $\frac{1}{10}$ rocket per tick in the white frame. You can (and should) convince yourself that this works for any other pairs of events.

I have described all this as if the clocks on both trains were deliberately set out of synchronization by the people in the station frame and indeed, if that is how the trains and clocks are set up, and if the people on either train are under the impression that their clocks were actually synchronized, then they will interpret their photographs exactly as we have done.

What is special about the world we live in is this: If the people in the station frame should chose to do the experiment for trains moving with a speed of u feet per nanosecond, and should they choose the clock asynchronization to be exactly u nanoseconds of disagreement per foot of rocket, then to set up the clocks on both trains all they would have to do was to furnish people on each train with a highly accurate set of clocks, set the trains into motion, and instruct the people on each train to synchronize their clocks. Nature herself would automatically provide the discrepancy between the station-frame interpretation of the clocks, and the interpretation from within each train.

Figure 1


(a)
:000 :002 :004 :006
0 1 2 3
3 2 1 0
:006 :004 :002 :000
(b)
:006 :008 :010 :012 :014
0 1 2 3 4
4 3 2 1 0
:014 :012 :010 :008 :006
(c)
:012 :014 :016 :018 :020 :022
0 1 2 3 4 5
5 4 3 2 1 0
:022 :020 :018 :016 :014 :012
(d)
:018 :020 :022 :024 :026 :028 :030
0 1 2 3 4 5 6
6 5 4 3 2 1 0
:030 :028 :026 :024 :022 :020 :018

10. Space-Time Diagrams

We have been describing various events taking place along a long straight railroad track (or along a long straight line of rockets) by representing the events as points on a blackboard or a piece of paper. Events taking place in the same place at the same time (coincident events) are represented by the same point.[1] In many of these figures we have taken a horizontal separation of two points to indicate a spatial separation of the events they represent, and a vertical separation to indicate a temporal separation. By exploring this kind of procedure a little more generally and systematically, it is possible to arrive at deeper — I would say, in fact, the deepest — understanding of what relativity has to tell us about the nature of space and time.

For simplicity we continue to deal with only one spatial dimension — all the events we shall consider take place along a single straight track.[2] Let us start with a particular frame of reference (Alice's) and specify some simple rules that Alice can use to specify events by points on a page. Until Bob appears on the scene everything that follows refers to Alice's frame of reference. When I talk about events happening in the same place (or at the same time) I mean at the same place (or at the same time) according to Alice.

Rule 1. Two or more events that happen at the same place *and* at the same time (space-time coincidences) are all represented by the same point.

Rule 2. Events happening at the same place (but not necessarily at the same time) are represented by points on a single straight line. (Figure 1.) The line is called a line of constant (or fixed) position (or place).[3] Alice is clearly free to orient one such line of constant position in any direction she chooses, since such a choice amounts to nothing more than appropriately orienting the page on which she draws her diagram.

Rule 3. Any two such lines of constant position representing various events that happen in two *different* places must be parallel. For if they were not parallel they would intersect somewhere, and their point of intersection would correspond to a single event that happened in two different places. But by definition an event is something that happens at a single place (and at a single time).

Rule 4. In analogy with the usual conventions of map makers (more precisely, those making maps of regions very small compared with the radius of the Earth) Alice takes the

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[1] If we wish to draw a picture representing an event we can't of course, make the picture as small as a geometric point; similarly if we wish to draw two pictures representing both of two coincident events, we try to make the pictures as close together as we can.

[2] Adding the other two dimensions — horizontal and vertical distance away from the track — can sometimes give further insight, but it makes it impossible to draw everything on a page or blackboard. We shall therefore continue to restrict our attention to a single spatial dimension.

[3] If you want a more compact term try *equiloc* or *isotop*. As far as I know neither term is actually used by anybody, but one of them should be.

distance on the page between two distinct lines of constant position to be proportional to the actual distance between the positions of the events they represent. The quantitative relation between distances in space and distances in the diagram is given by a scale–factor λ ("lambda"). Multiplication by λ converts the actual spatial distance between two events into the distance on the page between the lines of constant position on which the events lie in the diagram. For example if lines of constant position separated by one cm on the page corresponded to events at positions one km apart, then λ would be one centimeter to the kilometer, numerically 1/100,000. If we wish to distinguish Alice's scale factor from those of people using other frames of reference (and it turns out to be important to be able to do this) we can give it a subscript, calling it λ_A.

The next three rules (5–7) simply specify for location in time, what Rules 2–4 specify for location in space.

Rule 5. Events happening at the same time (but not necessarily in the same place) are represented by points on a single straight line. (Figure 2.) The line is called a line of constant (or fixed) time.[4]. Alice is free to orient one such line of constant time to make any angle she wishes with her lines of constant position (except 0°, as noted in Rule 8 below), since such a choice of direction amounts to nothing more than an appropriate stretching of the page on which she draws her diagram (considered only for this purpose to be made of rubber).

Rule 6. Any two different lines of constant time must be parallel. For if they were not parallel they would intersect somewhere, and their point of intersection would correspond to a single event happening at two different times. But by definition an event is something that happens at a single time (and at a single place).

Rule 7. Alice takes the distance on the page between two distinct lines of constant time to be proportional to the actual time interval between the times of the events they represent. We defer her choice of scale factor to Rule 9.

Rule 8. Any line of constant time intersects any line of constant position in precisely one point, which represents those events that happen precisely at *that* time and in *that* place. Consequently the common direction of all her lines of constant time, though it is otherwise Alice's to choose, cannot be the same as the common direction of all her lines of constant position. The two families of lines must cross at some non-zero angle θ ("theta").

Rule 9. It turns out to be extremely convenient for Alice to take the distance in the diagram between two lines of constant time representing events one nanosecond apart to be exactly the same as the distance in the diagram between two lines of constant position representing events one foot[5] apart. Putting it in terms of scale factors, the scale factor λ for lines of constant position (in centimeters of diagram per f) is numerically the same as

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[4] One could also call it an *equitemp* or an *isochron*

[5] Recall that the foot is defined in Physics 209 to be the distance light travels in vacuum in one nanosecond.

the scale factor λ for lines of constant time (in centimeters of diagram per ns.)

Rule 10. With the convention adopted in 9, it follows that another convenient scale factor, the distance μ ("mu") along any line of constant position associated with two events one ns apart, is also exactly the same as the distance along any line of constant time associated with two events one f apart. (See Figure 3.) This is a consequence of the elementary fact that when a pair of parallel lines intersects another pair of parallel lines separated by the same distance as the first pair (in this case the distance is just λ), then the parallelogram defined by the four points of intersection has four equal sides.[6] Note that the scale factor μ exceeds the scale factor λ unless Alice takes her lines of constant time perpendicular to her lines of constant position, in which case $\mu = \lambda$.

Both scale factors are useful. Often it is easiest to extract the time (or distance) between events from the distance between the lines of constant time (or position) on which they lie, in which case λ is the relevant scale factor. But sometimes one wants to extract the time (or distance) between events from their distance apart on a line of constant position (or time), in which case μ is the relevant one.

A particularly important collection of events for an object small enough to be considered to occupy just a single point of space at any moment of time, is the set of *all* events at which the object is present. The totality of all such events is represented by a continuous line in the diagram. This line, which represents the entire history of the object, is the *world line* or *space-time trajectory* of the object. For example an object stationary in Alice's frame of reference throughout its entire history is represented by the line of constant position associated with the place the object occupies. An object moving uniformly in Alice's frame of reference is represented by a straight line that is not parallel to any line of constant position, since the object is at different positions at different times. An object that is moving non-uniformly — for example back and forth — is represented by a wiggly line.

A particularly important world line is the space-time trajectory of a photon, or of any other object moving at the speed of one foot per nanosecond. Lines of constant position and constant time have a very simple relation to photon trajectories. Any two events on a photon trajectory must be as many feet apart in space as they are nanoseconds apart in time. So as a result of Rule 9, the two lines of constant position passing through those two events must be the same distance apart in the diagram as the two lines of constant time passing through those events. Since the photon trajectory is thus the diagonal of a rhombus formed by the two pairs of parallel lines, the trajectory bisects the angles at the two vertices it connects. (See Figure 4.) We have thus deduced an extremely important property of Alice's diagram:

[6] A parallelogram with four equal sides is called a rhombus. A rhombus has the elementary property that the lines connecting opposite vertices — called "diagonals" — bisect the angles at those vertices. This will turn out to be useful.

Rule 11. The angle that lines of constant position make with the trajectory of a photon must be the same as the angle that lines of constant time make with that trajectory. Putting it another way, *the two photon trajectories through the point of intersection of a line of constant position with a line of constant time, bisect the angles formed by the two lines.*

Since this rule applies to photons moving in either direction we have a second important deduction (Figure 5 makes it clear why this follows from Rule 11):

Rule 12. The trajectories of two photons moving in opposite directions are *perpendicular* to each other in the diagram. Even though Alice can freely chose the angle θ between her lines of constant time and constant position, the scale convention adopted in Rule 9 requires certain angles to be fixed: *the world lines of oppositely moving photons are necessarily perpendicular.*

Rule 13. Because of Rules 11 and 12, Alice can rotate her page so that the trajectories of two photons moving in opposite directions are symmetrically disposed about the vertical direction, tilted at 45 degrees to the right and left, with the times of the events on each photon trajectory increasing as one moves along the page in the upward direction. Because Alice's lines of constant time make the same angle with the photon trajectories as her lines of constant position, her lines of constant position will then tilt away from the vertical at the same angle (less than 45 degrees) that her lines of constant time tilt away from the horizontal. It is conventional always to orient a space-time diagram in this way, so that the vertical and horizontal directions bisect the right angles between the two families of photon trajectories, and so that lines of constant time higher in the diagram represent events occurring at later times.

The above rules completely determine the structure and orientation of the system of lines of constant time and position that Alice uses to locate events in space and time except for two choices still available to her:

(a) She is free to choose the scale factor λ — i.e. the distance on the page between two lines of constant position associated with places one f apart (which is also the distance on the page between two lines of constant time associated with events one ns apart.)

(b) She is free to choose the angle θ that her lines of constant time make with her lines of constant position or, equivalently, the angle $\frac{1}{2}\theta$ that both families of lines make with the photon lines. (Her choice of λ and θ together fix the alternative scale–factor μ.)

Alice's choice of scale depends, of course, on how big a page she has and on the spatio–temporal extent of the collection of events she wishes to represent in her diagram. Her choice of angle depends on what she (or we) wish to do with her diagram. If she is using it only for her own private purposes then a pleasingly symmetric choice is to take θ to be 90 degrees, so that her lines of constant position are vertical and her lines of constant time, horizontal. If, however, she (or we) wish to compare the space-time description of events that she reads from her diagram with the space-time description of those same events provided by other observers using one or more other frames of reference, then taking θ to be 90 degrees need not give the clearest picture. To see why we must consider the use to which Alice's diagram can be put by people who prefer to describe events using other frames of reference.

Bob, moving uniformly along the track with velocity v with respect to Alice, wishes to describe the same events that she has been describing, but prefers a frame of reference in which he is at rest. Suppose Bob is shown Alice's diagram, filled with points that represent isolated events and lines that represent space-time trajectories, but without any of her lines of constant time and position that she might have drawn to help her locate those events in space and time. Rather than make his own independent diagram to describe those various phenomena, Bob can use precisely the same collection of points and lines that Alice used. But he will describe them in a different spatio–temporal language, since he will disagree with Alice's general notions of "same place" and "same time". He will therefore not use the same lines of constant position and time that Alice uses. It is not hard to figure out what he must do.

If Bob's frame of reference has velocity v with respect to Alice's, then Bob's lines of constant position must be parallel to the space-time trajectory of a particle that Alice maintains is moving with velocity v. Thus Bob's lines of constant position are parallel straight lines that are not parallel to Alice's lines of constant position. The faster Bob moves with respect to Alice, the more they tilt away from Alice's lines. Lines of constant Alice-time and Alice-position through any two points on one of Bob's lines of constant position, define a parallelogram the ratio of whose sides (or the ratio of the distances between those sides) is just the velocity v of his frame with respect to hers. See Figure 6.

We can also determine the orientation of Bob's lines of constant time. We do this by putting into the diagram a set of events associated with a clock synchronization experiment, carried out on a train that is stationary in Bob's frame of reference. The left end, right end, and middle of such a train are represented in Alice's diagram by lines of constant Bob-position. Since Alice agrees with Bob about what point on the train constitutes its middle, the lines are equally spaced in Alice's diagram. Two photons created together in the middle of the train travel in opposite directions at the same speed. Since the train is stationary in Bob's frame, and both photons have the same speed in his frame (namely one f/ns — this where the invariance of the velocity of light enters the story) they arrive at the two ends of the train at the same Bob-time. So if we draw a pair of 45° lines that start at a point on the trajectory of the middle of the train, representing the trajectories of photons moving toward the front and rear, then the points of intersection of the two photons with the two ends of the train represent simultaneous events in Bob's frame and therefore lie on one of his lines of constant time. (Part (a) of Figure 7.)

It is then easy to deduce that Bob's lines of constant time in Alice's diagram must make the same angle with the photon trajectories as his lines of constant position do. One sees this most directly by letting each photon be reflected from its end of the train back to the middle. The resulting collection of photon trajectories (part (b) of Figure 7) form the four sides of a rectangle. It is evident (as explained in part (b) of the caption of Figure 7) that all the angles with the same label are equal. Therefore the photon trajectories passing through the black dot on the left do indeed bisect the angles between Bob's lines of constant position and time passing through that dot.

Note that this conclusion is identical to Rule 11 for the orientation of Alice's lines of constant time and position. Furthermore, because the common speed of both photons in Bob's frame continues to be 1 f/ns, two of Bob's lines of constant position associated with places one f apart in his frame, must be the same distance apart in Alice's diagram as two of his lines of constant time associated with times one ns apart in his frame.

Thus the rules we set up for the orientation of Alice's lines of constant time and constant position and the relation between their scales, impose restrictions on the lines of constant position and time that Bob must use, if he wishes to represent events with the same points that Alice uses in her diagram and, importantly, those restrictions turn out to have exactly the same form as the rules we originally imposed on Alice. It is therefore impossible for anybody else to tell which of them made the diagram first, following the rules described above, and which of them subsequently imposed his or her own lines of constant time and position on the other's diagram. This is of course required by the principle of relativity, but seeing it emerge in this way affords a vivid demonstration that the principle of relativity is indeed consistent with the frame independence of the velocity of light.

In summary, Alice and Bob (and Carol and Dick and Eve. . .) can all represent events in space and time by the same set of points in a single diagram, on which they each superimpose different families of lines of constant time and position. The lines in any one observer's family are symmetrically disposed about the two perpendicular directions along which photon trajectories lie — i.e. the photon trajectories through the point of intersection of lines of constant time and position belonging to a single frame of reference, bisect the angles between those two lines.

There remains the question of how people using different frames of reference relate their scale factors λ which give the distance on the page between their lines of constant position associated with events one f apart and between their lines of constant time associated with events one ns apart. One can acquire substantial insight from appropriately drawn space-time diagrams without ever needing the quantitative relation between scale factors, so I simply state here what the rule is:[7]

Call a rhombus bounded by lines of constant time and position associated with events one ns and one f apart a *unit rhombus*. The scale factors used by different frames of

[7] No use is made of this rule in the examples that follow.

reference are related by the rule[8] that *unit rhombi used by different observers all have the same area.* Since the altitude of a unit rhombus is the scale factor λ and its base is the scale factor μ (Figure 8), the analytical expression of this geometric rule is that for any two frames of reference

$$\lambda_A \mu_A = \lambda_B \mu_B. \tag{10.1}$$

Figures 9-11 show a few ways in which these space-time diagrams clarify some of the puzzles raised by relativity:

(1) Figure 9 shows how it is possible for each of two sticks in relative motion to be longer than the other in its proper frame. The two solid vertical lines represent the space-time trajectories of the left and right ends of the first stick. Lines of constant position in the proper frame of the first stick are vertical (since each end of the stick does not change its position in that frame), so lines of constant time in the proper frame of the first stick must be horizontal. Any horizontal slice of the figure shows what things are like at that given moment of time in the frame of the first stick.

The two parallel solid vertical lines that slant upward to the right represent the space-time trajectories of the left and right ends of the second stick. They constitute lines of constant position in the proper frame of the second stick. Lines of constant time in the proper frame of the second stick tilt away from the horizontal by as much as the lines of constant position tilt away from the vertical. Any slice of the figure with such a tilted line of constant time shows what things are like at that give moment of time in the frame of the second stick.

The horizontal dashed line in Figure 9 is one such line of constant time in the frame of the first stick. As you look along that line from left to right you encounter first the left end of the first stick, then the left end of the second, then the right end of the second, and finally the right end of the first. Thus in the proper frame of the first stick the two ends of the first stick extend beyond the two ends the second stick: the second stick is shorter than the first.

On the other hand the tilted dashed line is a line of constant time in the frame of the second stick. As you move along that line from lower left to upper right you encounter first the left end of the second stick, then the left end of the first, then the right end of the first, and finally the right end of the second. Thus in the proper frame of the second stick the two ends of the second stick extend beyond the two ends of the first stick: the first stick is shorter.

What the figure makes absolutely explicit is that if two sticks are in motion relative to one another, then their comparative lengths depend on the convention one employs for

[8] I show in the Appendix below that this rule follows directly from the requirement that when Alice and Bob move away from each other at constant velocity each must *see* the other's clock running slowly in the same way.

the simultaneity of events in different places. The various pieces of a stick (its two ends, its middle, a point two thirds of the way along the stick, etc.) are situated in different places. Which parts of the space-time trajectories of each piece of the stick one puts together to make up what one would like to call *the stick* at a given moment of time, depends on which events in the history of each of those pieces one chooses to regard as simultaneous. What is independent of any such convention, is the totality of all the space-time trajectories of all the pieces of both sticks. What is conventional and frame-dependent, is how one chooses to slice those trajectories with lines of constant time to form the *stick-at-a-given-moment.*

Note that there is also a frame of reference (moving to the right with respect to the first stick at a speed less than the second) in which both sticks have the same length. That frame is the one which has a line of constant time that can join the point of intersection of the trajectories of the left ends of the sticks with the point of intersection of the trajectories of the right ends.

(2) Figure 10 shows how it is possible for each of two clocks, in relative motion, to run faster than the other in its proper frame. The vertical row of numbered circles represents seven moments in the history of a clock and the reading of the clock (in seconds) at those moments. The slanting row represents six moments in the history of a second clock, moving to the right relative to the first and its reading (in seconds) at those moments. (Both clocks are in the same place at the same time when they read 0, and are therefore represented at that moment in their histories by one and the same circle.)

Lines of constant position in the proper frame of the first clock are vertical (since the line on which the seven moments in the history of the first clock lie is vertical) so lines of constant time in the proper frame of the first clock are horizontal. Since the second clock reads 4 and the first clock reads 5 on a horizontal line and since they both read 0 when they were together, the second clock is running at 4/5 the rate of the first, in the proper frame of the first clock.

Lines of constant position in the proper frame of the second clock have the same tilt as the line on which the six moments in the history of the second clock lie. Lines of constant time in the proper frame of the second clock make the same angle with the horizontal as lines of constant position make with the vertical. Such a line of constant time is shown connecting the moment when the first clock reads 4 and the second clock reads 5. Since they both read 0 when they were together, the first clock is running at 4/5 the rate of the second, in the proper frame of the second clock.

Figure 10 makes explicit the fact that a comparison of the rates of two clocks in relative motion depends crucially on the convention one adopts for the simultaneity of events in different places.

(3) If we stopped with Figure 10, which clock was *actually* running slower would be a matter of convention, empty of real content. Suppose, however, that the second clock

suddenly reverses its direction of motion and returns to the first. One can then compare them directly when they are back at the same place at the same time and see which has advanced by the greater amount.

In thinking about this it is important to recognize that the process of turning around breaks the symmetry between the two clocks. The first clock is stationary in a single inertial frame of reference throughout its entire history. The proper frame of the second clock, however, changes from one inertial frame of reference (moving uniformly to the right) to another (moving uniformly to the left) at the moment it turns around. There is no single inertial frame of reference in which the second clock is stationary throughout its history, and the enormous decelerations and accelerations attended upon turning around and heading back to earth will be quite evident to anybody moving with the second clock.

In the frame of reference of the first clock (which uses horizontal lines of constant time) it is clear from Figure 11 that when the trip is over the second clock will have advanced only by 8 (4 on the outward journey and 4 on the inward journey) while the first clock has advanced by 10. So when the two clocks come back together the first will read 10 and the second, 8, as indicated in the figure.

Things are trickier from the point of view of the second clock, since two different inertial frames of reference are involved. In the frame moving outward with the second clock, the first clock runs slowly and advances only by 3.2 (from reading 0 to reading 3.2) during the time the second advances by 4 (from 0 to 4). This is revealed by the lower of the two tilted lines of constant time in Figure 11. Similarly, in the frame moving inward with the second clock, the first clock is also running slowly and advances only by 3.2 (from reading 6.8 to reading 10) as the second advances by 4 (from 4 to 8), as revealed by the upper tilted line of constant time in Figure 11.

The indisputable fact that the first clock reads 10 and the second reads only 8 when they are reunited makes sense from the point of view of the second clock, even though the first clock runs slowly in both the outgoing and the incoming frames. The missing 3.6 units of first-clock time ($3.6 = 10 - 2 \times 3.2$) comes from a correction that must be made in the notion of *what-the-first-clock-reads-now* when the second clock changes frames. As Figure 11 shows, at the place and time of turn-around, when the second clock reads 4, the far-away first clock *now* reads 3.2 according to the notion of simultaneity in the outgoing frame, but it *now* reads 6.8 according to the notion of simultaneity in the incoming frame. It is this adjustment, with the change of frames, of what the first clock is doing *now*, that accounts for the missing time.

(4) The essential role played by the different simultaneity conventions in different frames of reference, drops out of the story if we ask not what people moving with each clock *say* about the current reading of the other clock, but what they actualy *see* it doing. Figure 12 reproduces the clocks of Figure 11, without the lines of constant time appropriate to the three different frames of reference, but with the trajectories (dotted lines) of photons

emitted by each clock as its reading changes. Since the slowing-down factor for the moving clocks is 4/5, the relative velocity of the clocks is $v = \frac{3}{5}c$, and therefore the Doppler factor,[9] $\sqrt{\frac{1+v/c}{1-v/c}}$ is 2: people watching a clock moving away from them (or moving away from the clock) at 3/5 the speed of light, will *see* it running at half its proper rate; people watching a clock moving towards them (or moving towards the clock) at 3/5 the speed of light, will *see* it running at twice its proper rate.

People with the first clock (which has the vertical line of constant position) see the light emitted by the second clock as it changes to 1, 2, 3, and 4, as the first clock reads 2, 4, 6, and 8; they see the light emitted by the second clock as it changes to 5, 6, 7, and 8 as the first clock reads 8.5, 9, 9.5, and 10. So they see the second clock running at half its proper rate for 80% of the time and at twice its proper rate for 20% of the time. The considerable time seen running slowly overwhelms the rather brief time seen running fast, and the net effect is that the second clock has not advanced as much as the first when the journey is over.

On the other hand people with the second clock (which has the slanting lines of constant position) see the light emitted by the first clock as it changes to 1 and 2, as the second clock reads 2 and 4. They see the light emitted by the first clock as it changes to 3, 4, 5, 6, 7, 8, 9, and 10, as the second clock reads 4.5, 5, 5.5, 6, 6.5, 7. 7.5, and 8. So they see it running at half its proper rate for half the time and at twice its proper rate for half the time. Since it is seen running at twice its proper rate for half the time, this already insures that it will have advanced by as much as the second clock when they are back together, and because it is seen running at half its proper rate for the remaining half, it will have advanced by 25% more than that when they are back together.

Appendix: Scale factors and the invariance of the interval.

We first establish the connection between the scales Alice and Bob use on their lines of constant position (or lines of constant time). A segment of a line of constant Alice position separating events a time T apart in her frame, is related to a segment of a line of constant Bob position separating events a time T apart in his, by the following rule (illustrated in Figure 13 below): *The rectangles of photon trajectories having the segments for diagonals have the same area.*

This rule is illustrated in Figure 13. Part (a) of Figure 13 shows two the two moments at which a clock, stationary in Alice's frame, reads 0 and T. The two moments in the history of the clock lie on a line of constant Alice position a distance $\mu_A T$ apart. The two photon trajectories emerging from the lower picture of the clock and the two entering the upper picture of the clock form a rectangle which has as its diagonal the segment of Alice's line of constant time connecting the clocks. Part (b) of Figure 13 shows the same state of

[9] This was discussed in Lecture Notes #7. Note that the factors of 2 and $\frac{1}{2}$ emerge automatically from the geometry of Figure 12.

affairs for a clock stationary in Bob's frame. The length $\mu_B T$ of the segment of Bob's line of constant position connecting the two moments in the history of his clock exceeds the length $\mu_A T$ of the corresponding segment associated with Alice's line of constant position. But it can be shown that the areas of the two surrounding dashed rectangles are exactly the same.

To see why, take the case in which the two clocks are in the same place when they read 0. This is illustrated in Figure 14.[10] Let an observer moving with Alice's clock *look at* Bob's clock at the moment Alice's reads T, and let an observer moving with Bob's clock *look at* Alice's at the moment Bob's reads T. Each will see the other's clock reading *the same* earlier time t.[11] A glance at the figure reveals that the ratio h/H of the short side of Bob's rectangle to the short side of Alice's, is the same as the ratio[12] t/T as determined from Alice's line of constant position, while the ratio b/B of the long side of Alice's rectangle to the long side of Bob's, is the same as the ratio[13] t/T as determined from Bob's line of constant position. But if $h/H = b/B$ then

$$hB = bH. \tag{10.2}$$

The left side of (10.2) is the area of Bob's rectangle (look at the figure!) and the right side is the area of Alice's. This is what we wished to establish.

That the equality of the rectangles leads immediately to the equality of the product $\lambda\mu$ of scale factors follows from the fact that four copies of either of the two identical triangles making up either rectangle (Part (a) of Figure 15) can be reassembled into a rhombus whose sides have length μT and are a distance λT apart (Part (b) of Figure 15). The area of the rhombus is $\lambda\mu T^2$, so the area of the rectangle is $\frac{1}{2}\lambda\mu T^2$ (where one uses λ_A and μ_A for Alice's rectangle and λ_B and μ_B for Bob's. Since Alice's and Bob's rectangles have the same area, this establishes (1) that the product $\lambda\mu$ is independent of frame of reference and (2) that the relation between the area A of either rectangle in Figure 13 and the time T between the clock present at the two events on its opposite corners is just

$$T^2 = A/(\tfrac{1}{2}\lambda\mu). \tag{10.3}$$

Abstracting from this, we conclude that the area of a rectangle of photon trajectories with two time-like separated events at opposite vertices, is just a frame-independent scale

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[10] Figure 14 results from simply sliding (without rotating) part (b) of Figure 13 over to part (a), to bring the two clocks reading 0 into coincidence, and then adding a few things.

[11] Each will see the same time, because each looks after the same time (T) has passed on his or her own clock, and each regards the other's clock as moving away at the same speed — i.e. the relation between Alice, Bob, and their clocks is completely symmetric.

[12] Actually, the ratio $\mu_A t/\mu_A T$, but the common scale factor μ_A has no effect on the ratio.

[13] Actually $\mu_B t/\mu_B T$.

factor $(\frac{1}{2}\lambda\mu)$ times the square of the time between the two events in the frame in which they happen at the same place.[14] This is precisely the *squared interval* I^2 between the events:

$$I^2 = A/(\tfrac{1}{2}\lambda\mu). \tag{10.4}$$

Finally one can see directly from Figure 16 that the squared interval between two time-like separated events is indeed the difference of the square of the time between them and the square of the distance between them, regardless of the frame in which that time and distance are evaluated:

The two events are the two large black circles. The thin solid line emerging from the lower event is a line of constant position in Carol's frame, and the thin solid line emerging from the upper event is a line of constant time in Carol's frame. The thin photon lines complete these thin solid lines to triangles. The thick solid line connecting the two events and the two thick dashed photon lines passing through the events form a third right triangle. If T is the time and D the distance between the events in Carol's frame then the geometric condition that $I^2 = T^2 - D^2$ is just that the area of the thick-sided triangle is equal to the area of the larger thin-sided triangle minus the area of the smaller one.[15]

The relation between the areas follows from the fact that the two thin-sided triangles differ only by a scale factor[16] γ ("gamma"). As a result, if the two sides of the bigger of the thin triangles are a and b, then the corresponding sides of the smaller of the thin triangles are γa and γb, and the two sides of the thick triangle are $(1-\gamma)a$ and $(1+\gamma)b$. The difference in areas of the thin triangles is thus $\frac{1}{2}ab - \frac{1}{2}(\gamma a)(\gamma b) = \frac{1}{2}ab(1-\gamma^2)$, while the area of the thick triangle is $\frac{1}{2}(1-\gamma)a(1+\gamma)b = \frac{1}{2}ab(1-\gamma^2)$.

[14] Because of the explicit symmetry of the diagrams under the interchange of space and time we can also conclude that the area of the rectangle of photon trajectories with two space-like separated events at opposite vertices, is that same invariant scale factor times the square of the distance between the two events in the frame in which they happen at the same time.

[15] For each of these triangles is half of a rectangle whose respective areas are $\frac{1}{2}\lambda\mu I^2$, $\frac{1}{2}\lambda\mu T^2$, and $\frac{1}{2}\lambda\mu D^2$.

[16] This is because the thin photon trajectory on the upper left makes the same angle α with Carol's line of constant time as it does with her line of constant position.

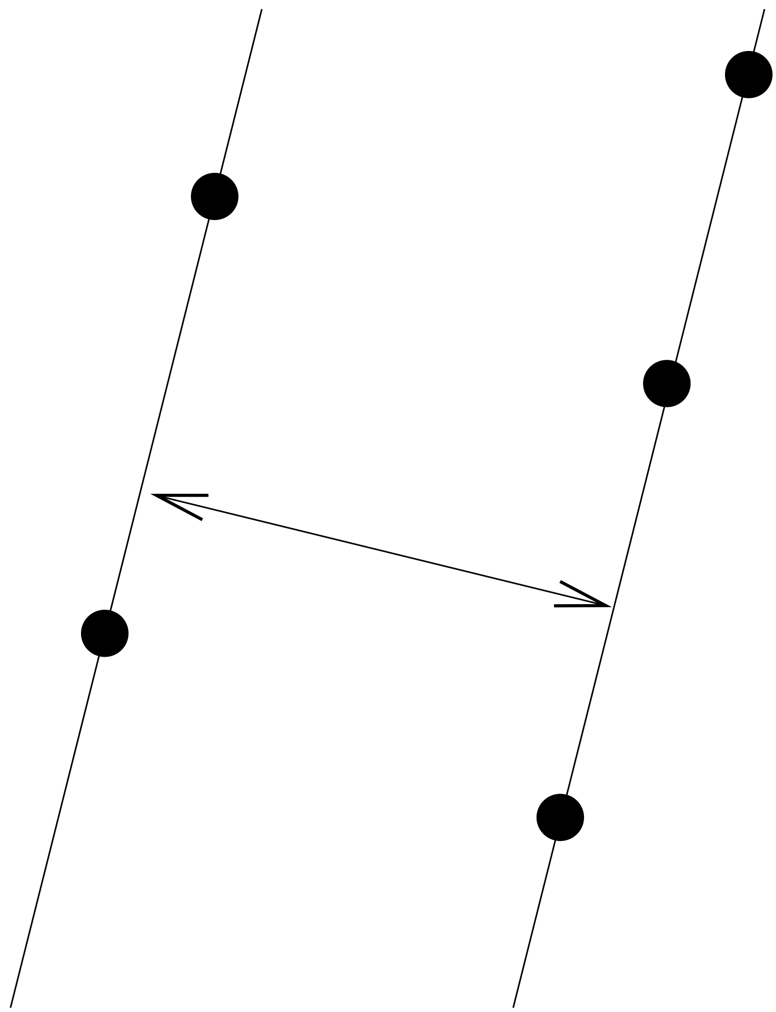

Figure 1. Two lines of constant position in Alice's frame. The two black dots on the line on the left represent two events that happen in a single place (but at different times) according to Alice; the three black dots on the line on the right represent three other events that happen in a single place, different from the place of the two events on the left. The distance between two such lines in the diagram (indicated by the double-headed arrow) is proportional to the actual distance in Alice's frame between the two places they represent. Such a diagram is characterized by a scale factor λ which specifies, for example, the number of centimeters on the page between lines representing places a foot apart in space.

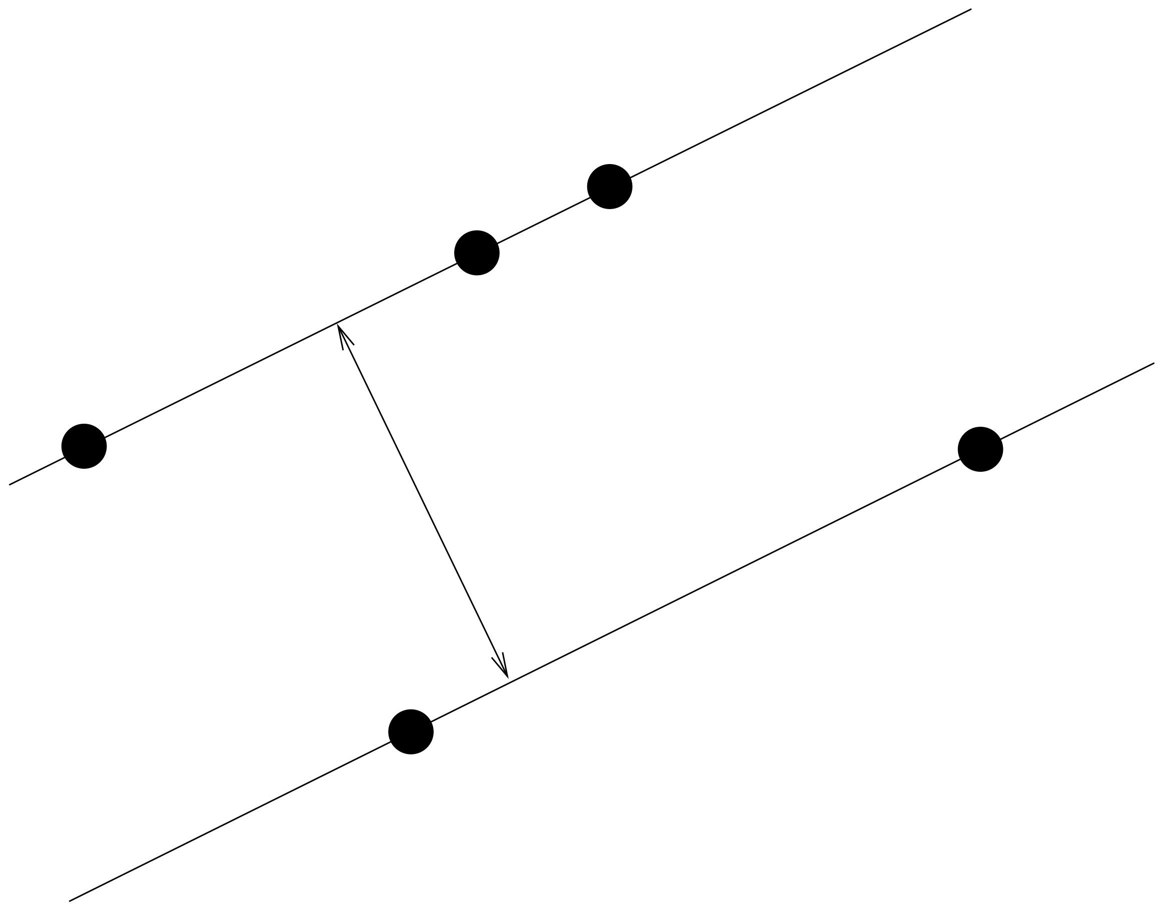

Figure 2. Two lines of constant time in Alice's frame. The two black dots on the lower line represent two events that happen in at a single time (but in different places) according to Alice; the three black dots on the upper line represent three other events that happen at a single time, different from the time of the two on the lower line. The distance between two such lines in the diagram (indicated by the double-headed arrow) is proportional to the actual time in Alice's frame between the two moments of time they represent. Such a diagram is characterized by a scale factor λ which specifies, for example, the number of centimeters on the page between lines representing events a nanosecond apart in time.

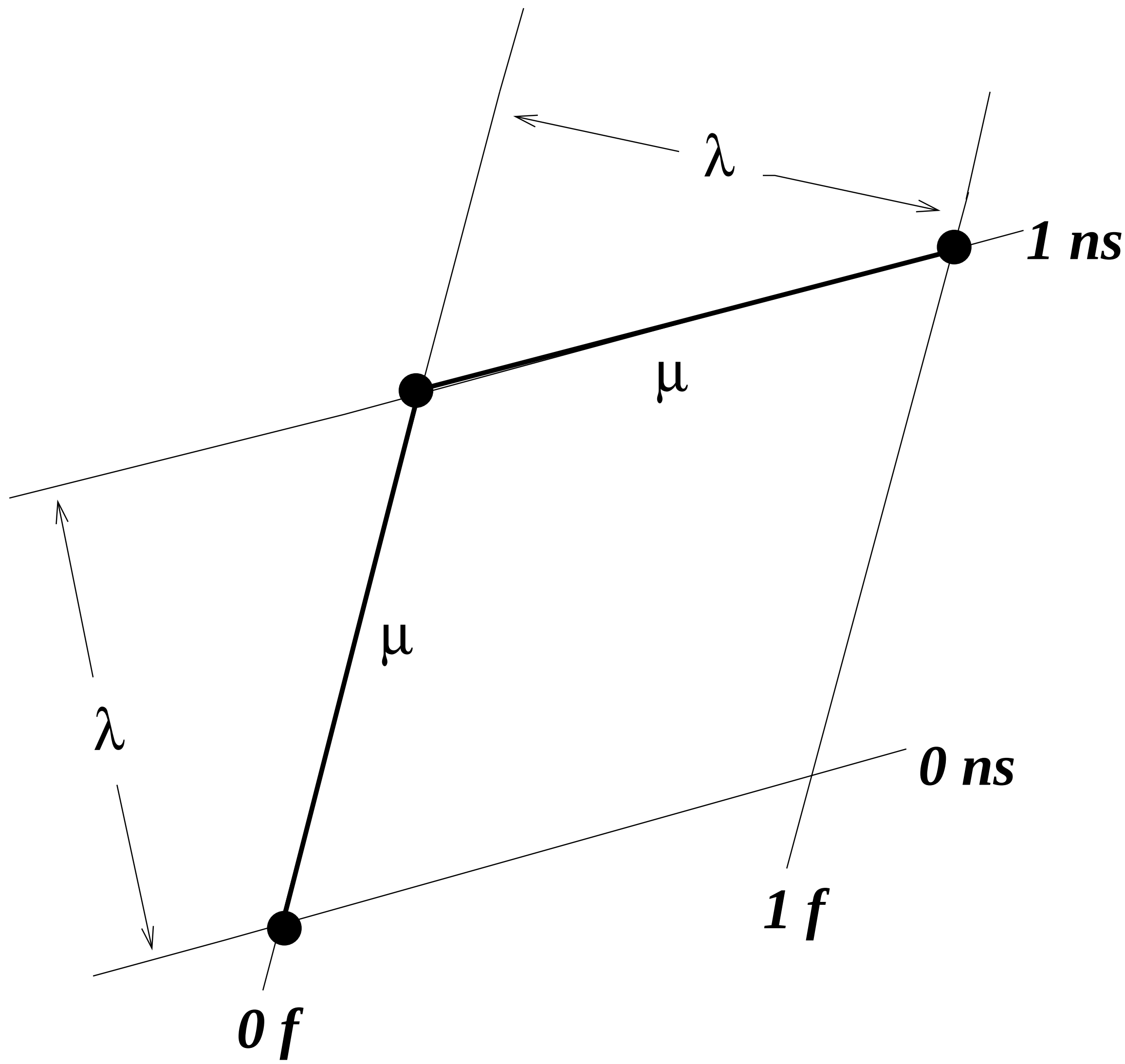

Figure 3. The scale factors λ and μ. The parallel lines tilting slightly upward to the right are lines of constant time; events represented by points on the upper line happen one nanosecond after events represented by points on the lower. The parallel lines tilting steeply upward to the right are lines of constant position; events represented by points on one line happen one foot away from events represented by points on the other. The scale factor λ is the distance in the diagram between the lines of constant position or between the lines of constant time. The scale factor μ is the length in the diagram of the (more heavily drawn) segments of the lines of constant time and position between the events represented by the black dots.

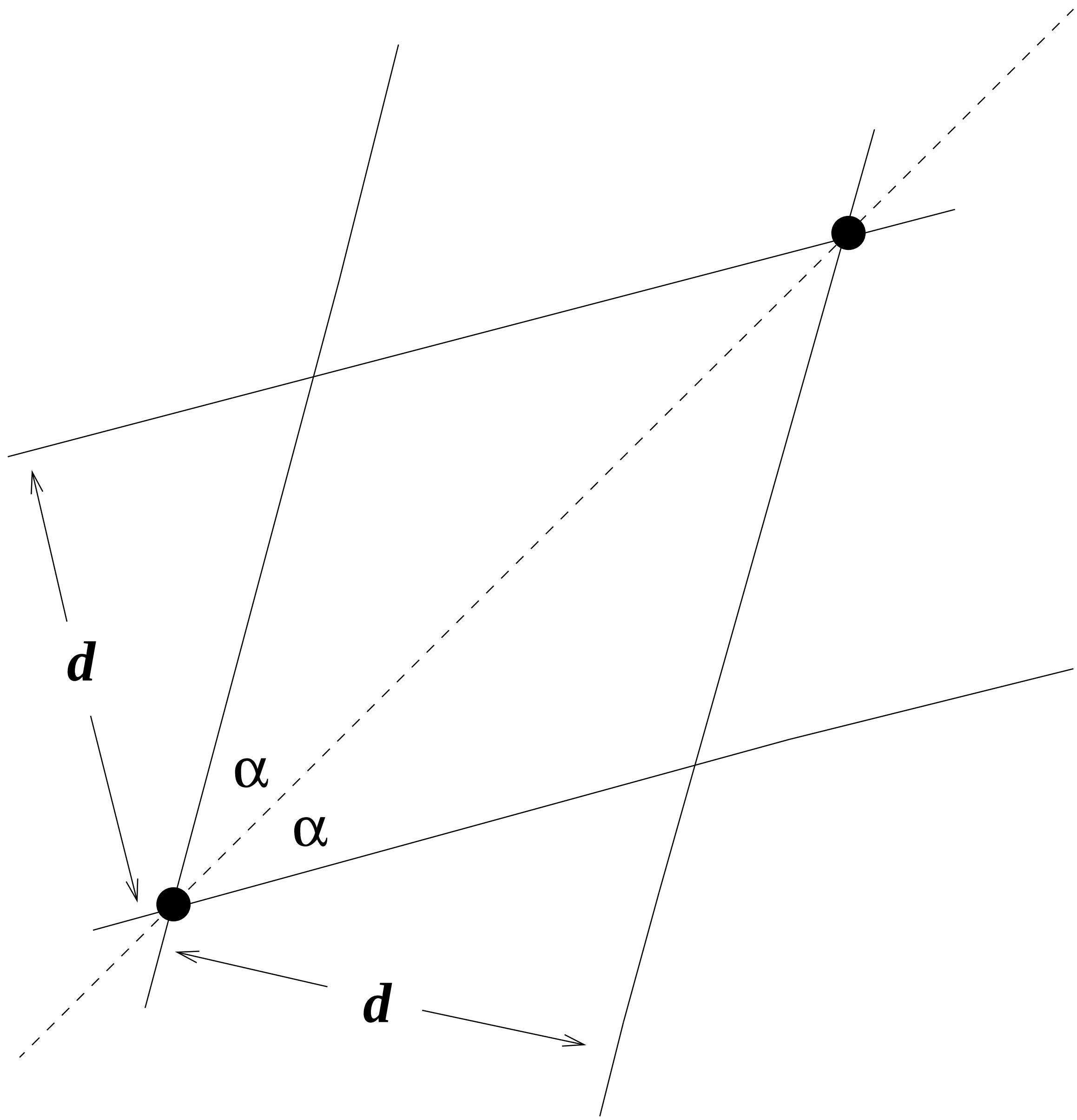

Figure 4. The dashed line represents the space–time trajectory of a photon. The black dots represent two events in the history of that photon. A line of constant position slants steeply upward through each dot, and a line of constant time slants slightly upward through each. Because the photon moves one foot every nanosecond, the distance d between the lines of constant position is the same as the distance d between the lines of constant time. The parallelogram formed by the four lines is therefore a rhombus, the dashed photon line is the diagonal of that rhombus, and the angles labeled α are equal.

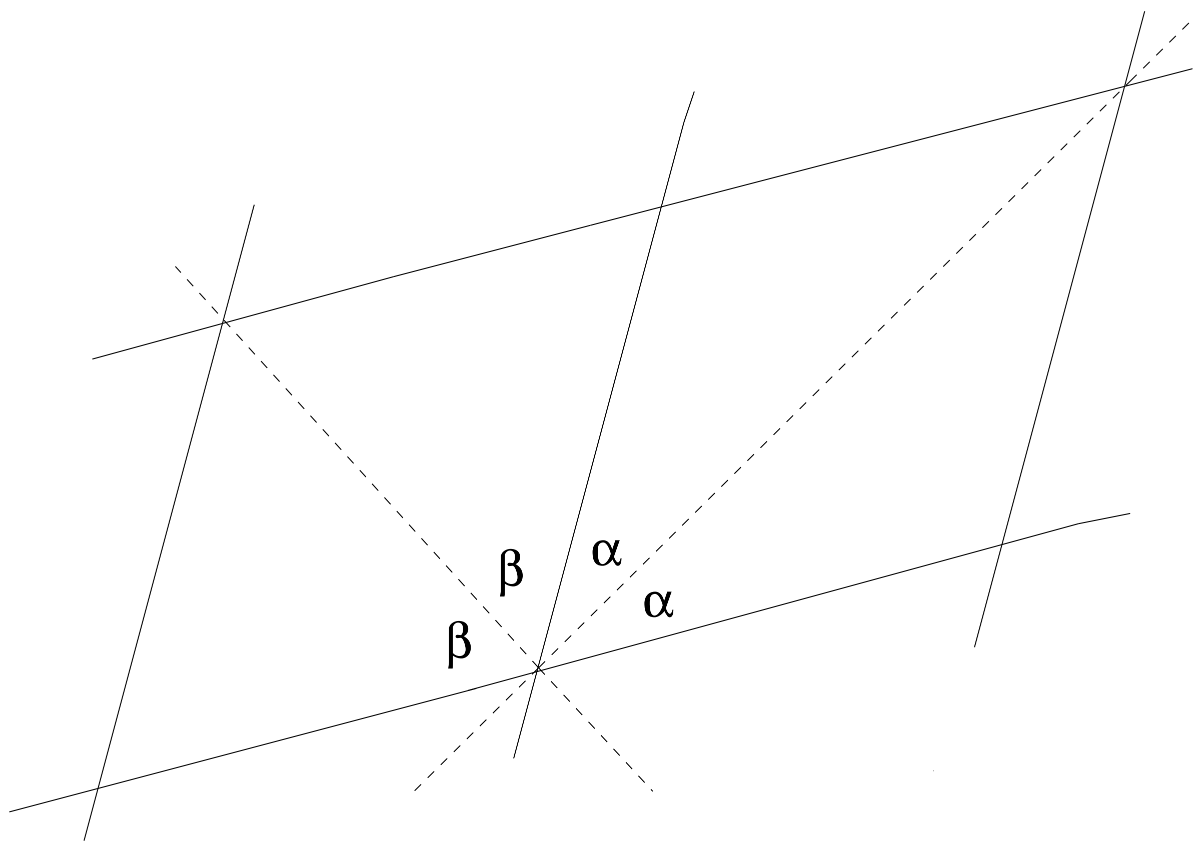

Figure 5. Figure 4 is redrawn (without the black dots) and extended to show the space–time trajectory of a second photon travelling in the opposite direction. Because the new dashed line is also a photon trajectory it also bisects the angle between the lines of constant time and position. Since $2\alpha + 2\beta = 180°$, the angle $\alpha + \beta$ between the two photon lines is $90°$.

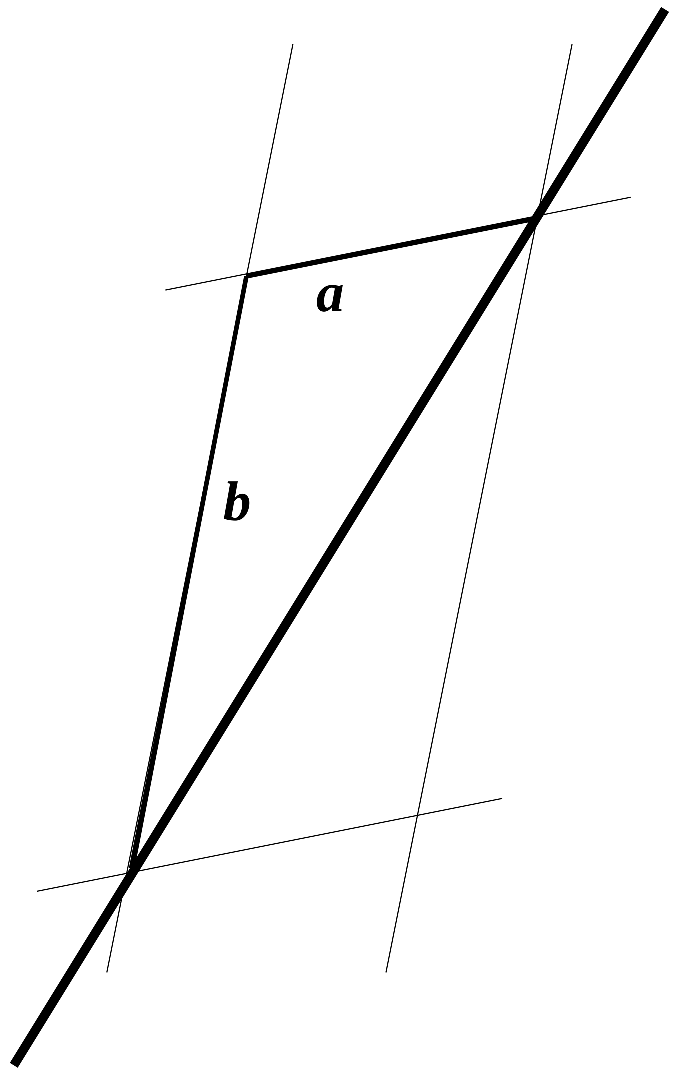

Figure 6. The very heavy line is the space-time trajectory of an object stationary in Bob's frame of reference — i.e. a line of constant position, according to Bob. The lighter lines are lines of constant Alice-time and Alice-position drawn through two events on the heavy trajectory. The lengths in the diagram of the darkened segments of those lines are a and b. The velocity of Bob with respect to Alice is $v = a/b$, since in Alice's frame the position of the object changes by a distance μa in a time μb. Note that a/b is also the ratio of the distance between the lines of constant Alice–position to the distance between the lines of constant Alice–time.

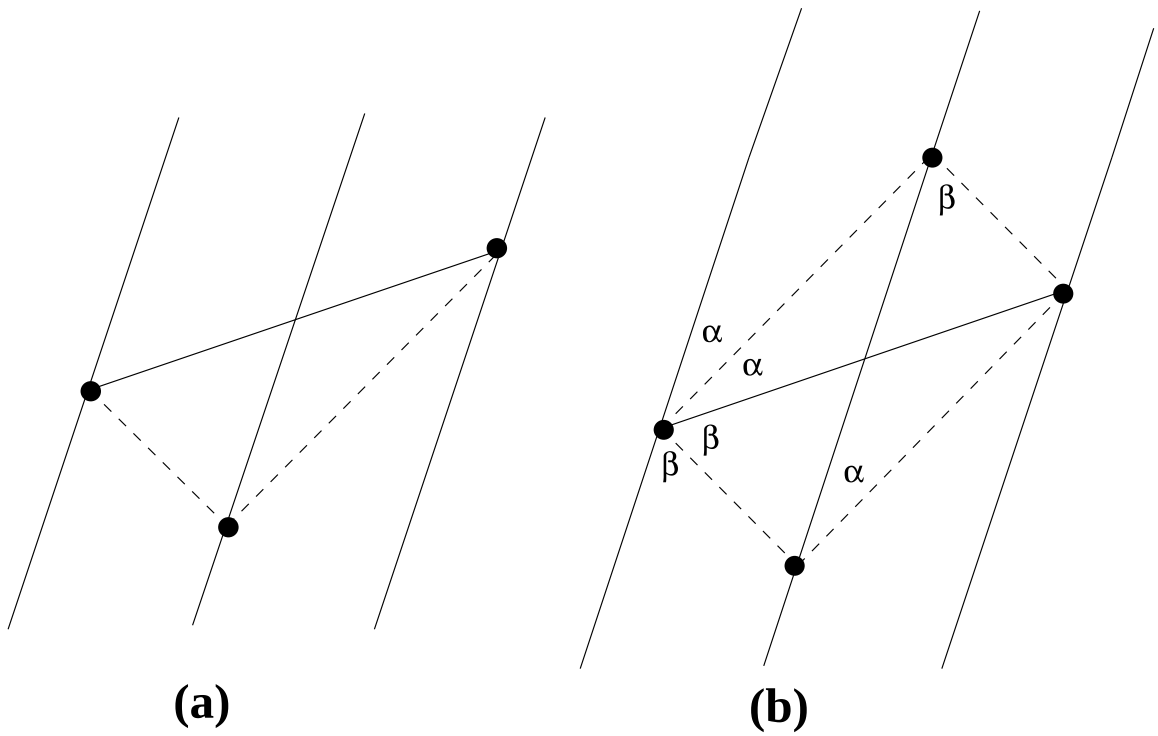

Figure 7. The diagram is drawn by Alice. (a) The three equally spaced parallel lines of constant position are the two ends and the middle of a train that is stationary in Bob's frame of reference. They establish the direction Bob's lines of constant position must have in Alice's diagram. The lowest black dot represents the production of two oppositely directed photons at the middle of the train. The dashed lines are the space-time trajectories of the photons. The other two black dots represent the arrival of each photon at an end of the train. Since both photons move at the same speed in Bob's frame of reference and since the train is stationary in Bob's frame, the photons arrive at the ends of the train at the same Bob–time — i.e. the line joining the upper two dots is a line of constant time in Bob's frame. (b) If the photons are reflected back toward the center of the train when they reach the two ends, they will arrive there at the same time in the event represented by the highest black dot, the four photon lines forming a rectangle. It is evident from the symmetry of the rectangle that the two angles labled α within the rectangle are equal, and so are the two labeled β. Since the two labeled angles outside the rectangle are just spatial translations of two correspondingly labeled angles within it, it follows that the photon trajectories passing through the left–most black dot bisect the lines of constant Bob-time and Bob-position passing through that dot.

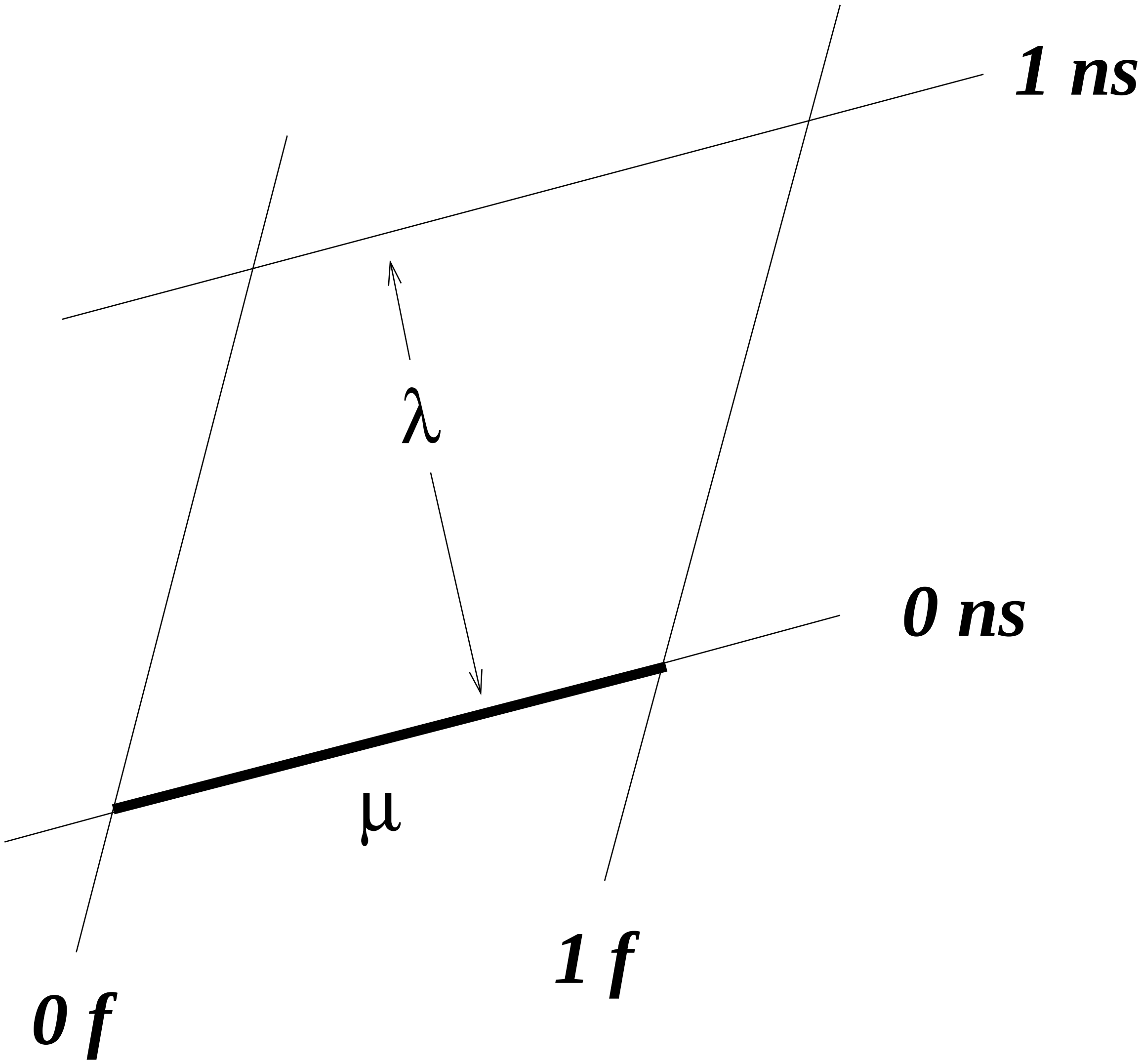

Figure 8. The unit rhombus for some frame of reference. The lines labeled 0 ns and 1 ns represent events one nanosecond apart and the lines labeled 0 f and 1 f represent events one foot apart. Because the distance in the diagram between the two lines of constant time — regarded as the height of the rhombus — is the scale factor λ and the heavier portion of the lower line of constant position — regarded as the base of the rhombus — is the scale factor μ, the area of the rhombus — its base times its height — is just the product $\lambda\mu$.

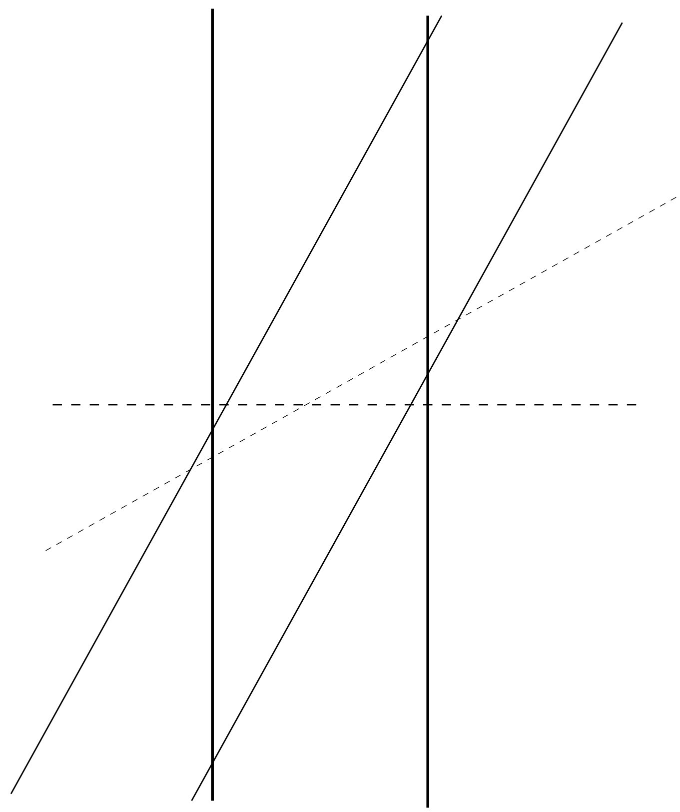

Figure 9. The two vertical solid lines are the left and right ends of a stick. The two solid lines that tilt upward to the right are the left and right ends of a second stick that moves to the right past the first stick. The horizontal dashed line is a line of constant time in the frame in which the first stick is stationary. Note that both ends of the first stick extend beyond both ends of the second along that line of constant time, thereby establishing that the first stick is longer than the second in its proper frame. The dashed line that tilts upward to the right is a line of constant time in the frame in which the second stick is stationary. (Note that it tilts away from the horizontal by the same amount that the lines representing the ends of the second stick tilt away from the vertical.) Along this tilted line of constant time both ends of the second stick extend beyond both ends of the first stick, thereby establishing that the second stick is longer than the first in its proper frame.

The figure vividly demonstrates that what one means by *a stick at a given moment of time* depends on the frame of reference in which the stick is described, and that it is this that makes it possible for people using the proper frame of either stick to maintain that the other stick is shorter.

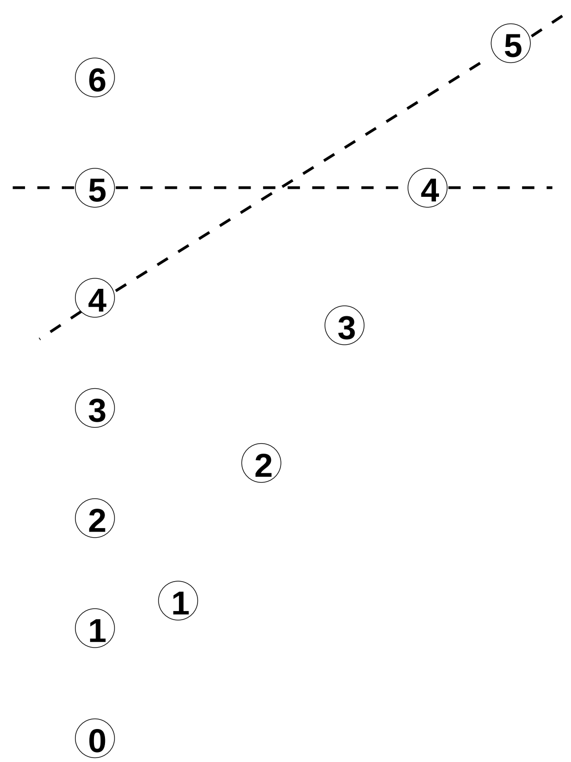

Figure 10. Several moments in the histories of two uniformly moving clocks (represented by circles with numbers inside giving their readings.) Both clocks read 0 at the same place and time and are represented by just a single circle in the figure. Subsequent readings of the first clock (1-6) are shown on the set of circles uniformly spaced along a vertical line; subsequent readings of the second clock (1-5) are shown on the set of circles that lie on a line sloping upward to the right. The horizontal dashed line is a line of constant time in the frame of the first clock. In that frame the second clock has advanced from 0 to 4 in the time it took the first to advance from 0 to 5, so the second clock is running slowly by a factor $s = 4/5$. The slanting dashed line is a line of constant time in the frame of the second clock (and tilts away from the horizontal by the same amount that the line along which the pictures of the second clock lie tilts away from the vertical.) In that frame the first clock has advanced from 0 to 4 in the time it took the second to advance from 0 to 5.

The figure makes clear that how one compares the rates of two clocks in relative motion depends on how one judges whether two events in different places are simultaneous. This is what makes it possible for people using the proper frame of either clock to maintain that the other clock is running slowly.

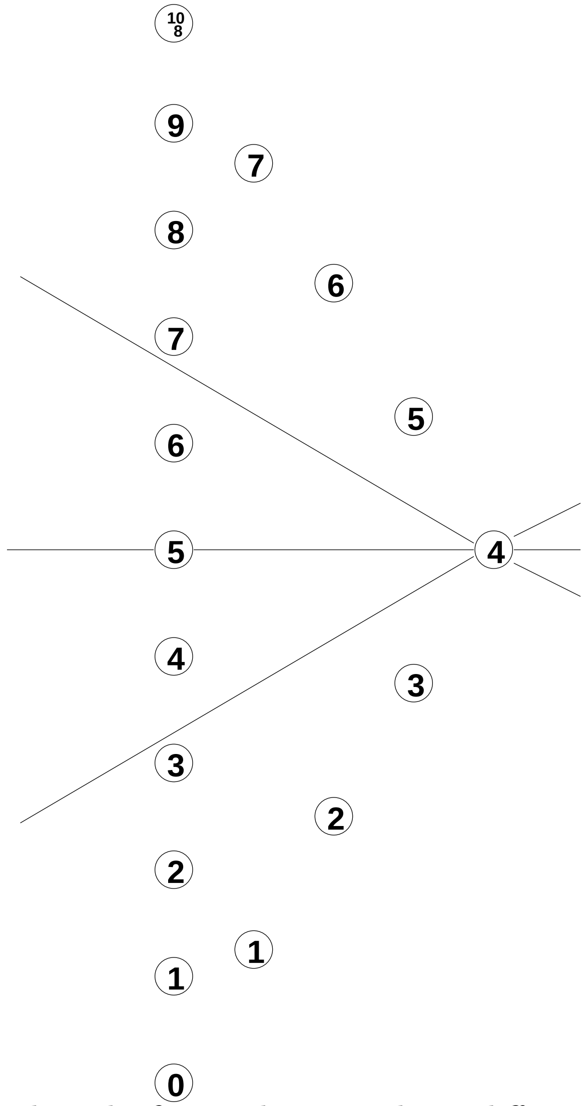

Figure 11. Two identical clocks. The first is shown at eleven different moments along the vertical line, as its reading advances from 0 to 10. The second moves away from the first as it advances from 0 to 4; it then moves back to the first, as its reading advances from 4 to 8. At the bottom and top of the figure both clocks are at the same place at the same time and are represented by a single circle. The first clock is stationary in a single inertial frame of reference. Since lines of constant position are vertical in that frame, lines of constant time are horizontal. Consequently it is evident from the figure that in the proper frame of the first clock, the outward and inward journeys of the second clock each take 5 seconds, during each of which the second clock only advances by 4 seconds.

Two other lines of constant time are shown passing through the point at which the second clock begins its return journey. One line (going downward to the left) is appropriate to the proper frame of the second clock during its outward journey; the other (going upward to the left) is appropriate to the proper frame of the second clock during its inward journey. Note that just before the second clock changes frames, the time on earth in the outward-going frame is about 3.2 seconds. Just after the second clock has changed frames, the time on earth in the inward-going frame is about 6.8 seconds.

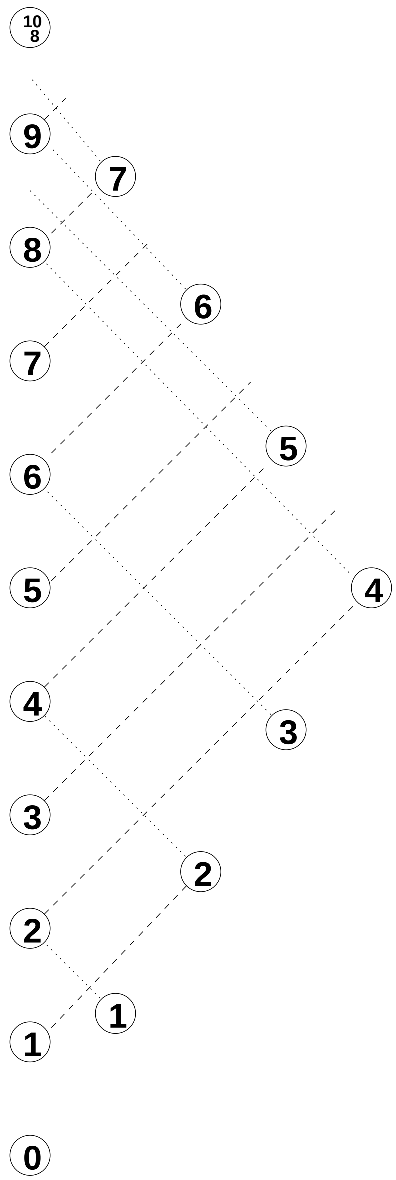

Figure 12. This repeats figure 9, but without the lines of constant time, and with many photon trajectories indicating what somebody moving with either clock *sees* the other clock doing. Each clock emits a flash of light each time its reading changes, and those flashes are seen by people moving with the other clock. People at the position of the first clock (vertical line of constant position) see the second clock advancing at *half* its normal rate during the 8 seconds they are watching the flashes emitted on the outward journey, and they see the second clock advancing at *twice* its normal rate during the 2 seconds they are watching flashes emitted by the second clock on its inward journey.

People moving with the second clock, on the other hand, see the first clock running at *half* its normal rate during the four seconds of their outward journey and at *twice* its normal rate during the four seconds of the inward journey.

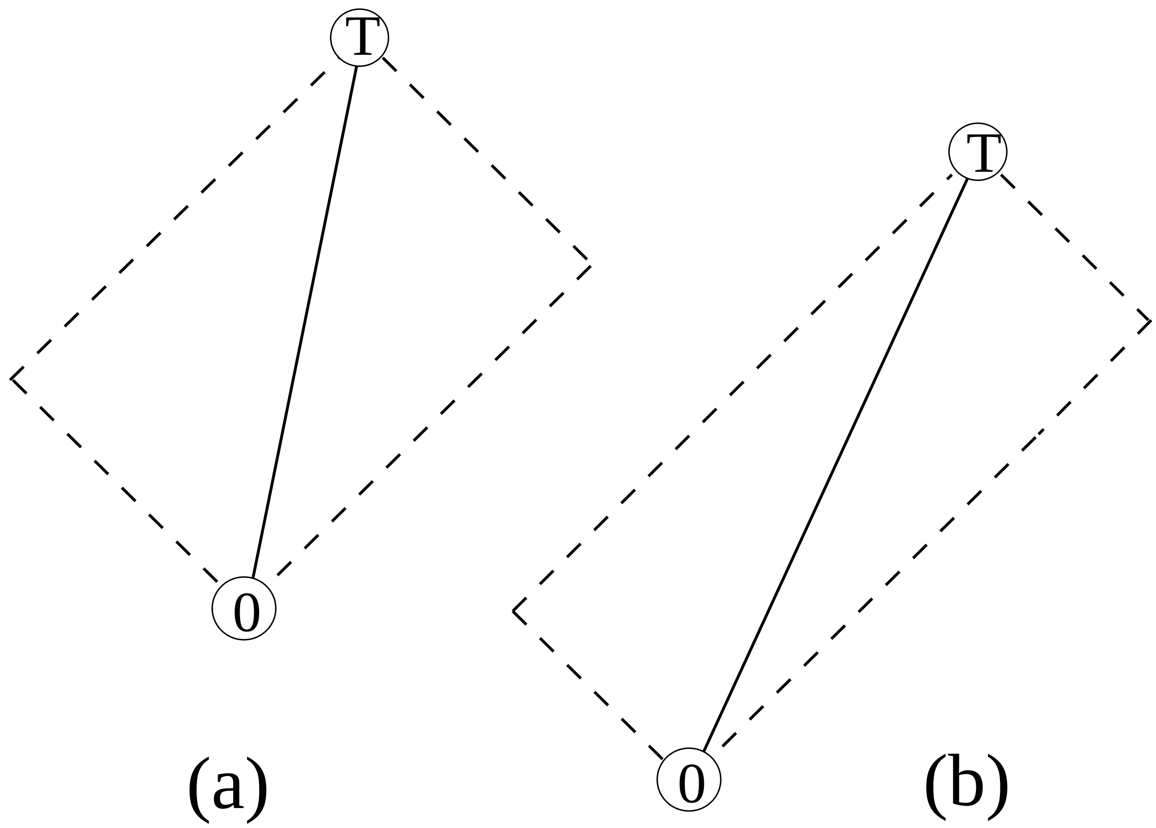

Figure 13. (a) A line of constant position in Alice's frame of reference, separating two events that are a time T apart in Alice's frame. The line can be viewed as the space-time trajectory of a clock that reads 0 at the first event and T at the second. (b) The same as (a), but for Bob's frame of reference. Note that the line connecting the events in which the clock stationary in Bob's frame reads 0 and T (which has length $\mu_B T$) is longer than the corresponding line in Alice's frame (which has length $\mu_A T$ — i.e. Alice and Bob use different scale factors μ to relate relate separation in time to distance along lines of constant position. However the *areas* of the two rectangles formed by photon trajectories emerging from the events are the same. This is established in Figure 14.

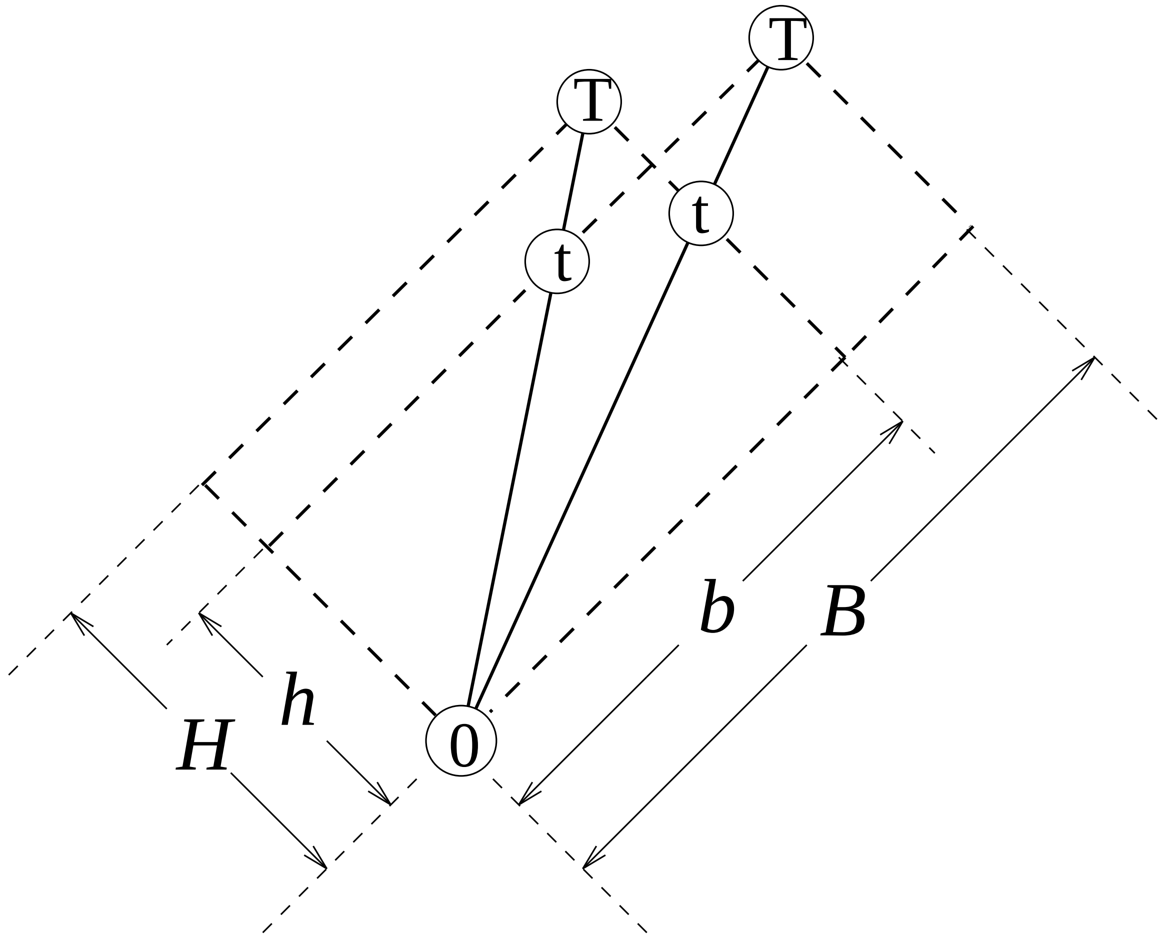

Figure 14. The two parts of figure 1 have been slid together (without rotating either) so that the two clocks reading 0 happen at the same place and at the same time. At the moment each clock reads T, somebody with the clock looks at the other clock and sees it reading t. The ratio of t and T along either line of constant position is just the ratio of the distances μt and μT from that moment in the history of the clock back to the moment at which it reads 0. (One uses μ_A for Alice's line and μ_B for Bob's.) It is evident from the figure that this ratio is also the same as the ratio of h to H or the ratio of b to B. But if $h/H = b/B$ then $Bh = bH$ — i.e. the two rectangles transported from Figure 13 to Figure 14 have the same area.

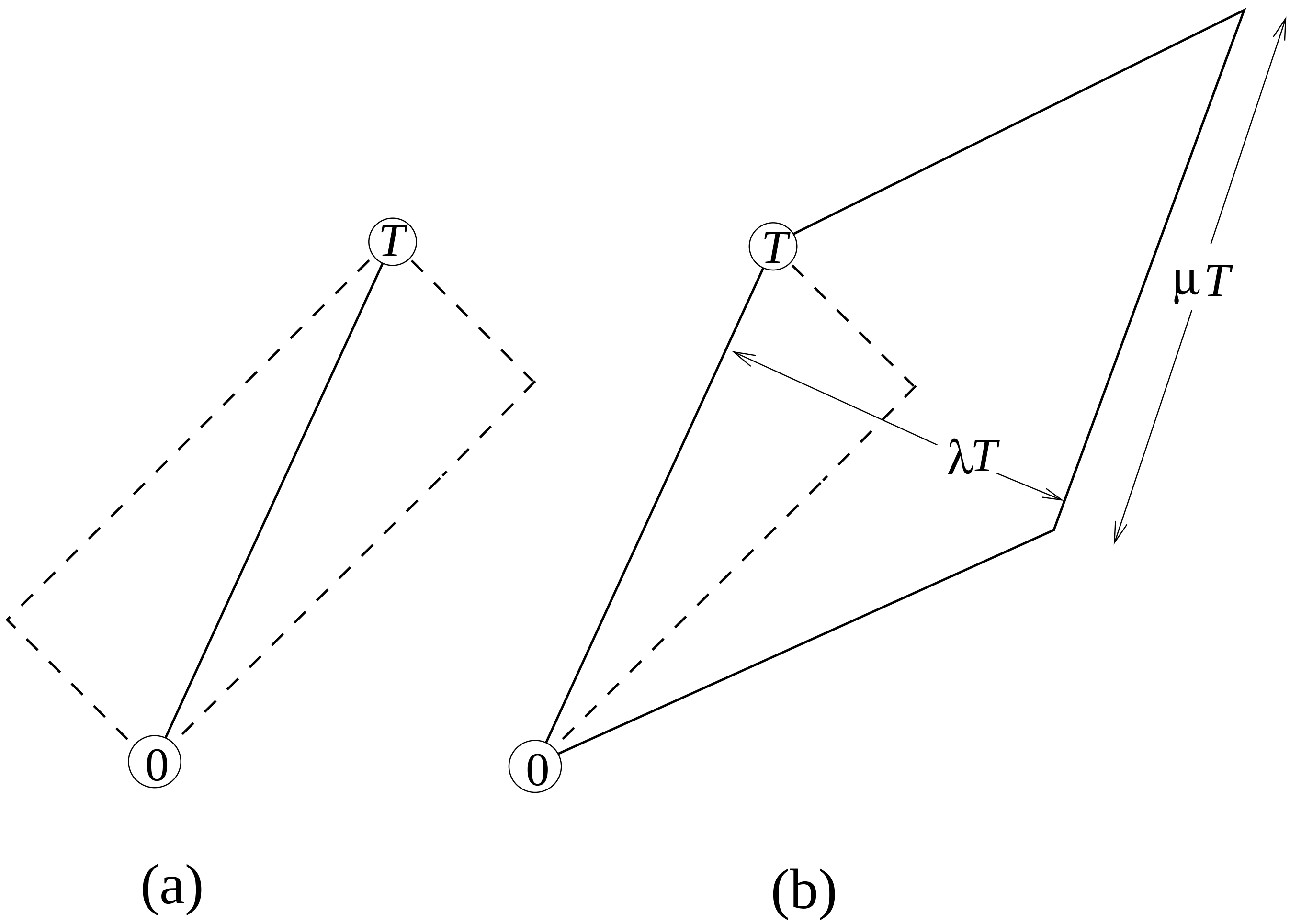

Figure 15. The area of either of the photon rectangles in Figure 13, shown here in part (a), is half the area of the rhombus shown in in part (b). (For the rhombus can be assembled out of four of the right triangles, two of which make up the rectangle.) But the area of the rhombus in part (b) is the length μT of a side times the distance λT between sides. So the area of the rectangle in part (a) is $\frac{1}{2}\lambda\mu T^2$.

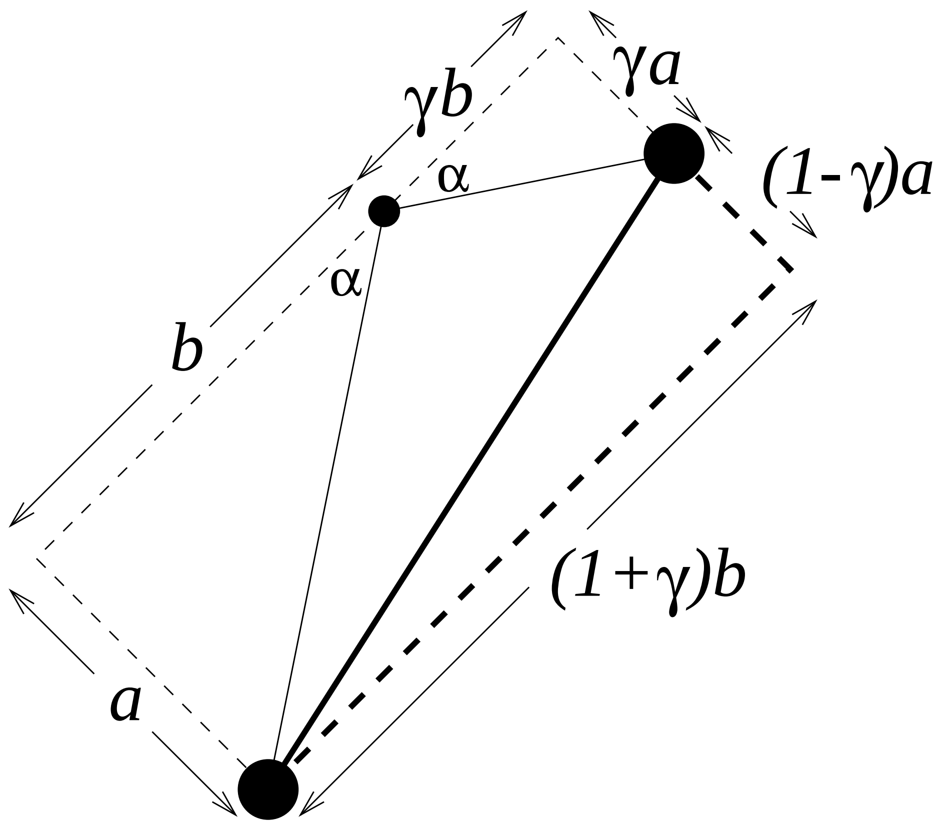

Figure 16. The two large black dots are two events. The two thin solid lines are Carol's lines of constant time and constant position. The difference between the areas of the thin-sided right triangles formed by those lines and the thin photon lines, which is clearly $\frac{1}{2}ba$ - $\frac{1}{2}(\gamma b)(\gamma a)$, turns out to be equal to the area of the thick-sided right triangle formed by the thick solid line joining the events and the thick photon lines, which is clearly $\frac{1}{2}([1+\gamma]b)([1-\gamma]a)$. (This equality is not geometrically obvious, but it follows immediately from the above forms and the algebraic identity $[1+\gamma][1-\gamma] = 1-\gamma^2$.)

Since the area of the thick-sided triangle is $\frac{1}{4}\mu\lambda$ times the squared interval, I^2 between the events, the area of the large thin-sided triangle is $\frac{1}{4}\mu\lambda$ times the square of the time T between the events in Carol's frame, and the area of the small thin-sided triangle is $\frac{1}{4}\mu\lambda$ times the square of the distance D between the events in Carol's frame, this establishes that $I^2 = T^2 - D^2$.

Note: one could make the same point using rectangles (like those in Figure 13) rather than one of the two right triangles that makes up each such rectangle. However Figure 16 would be more cluttered and the geometry of the relationship less evident, if each of the three right triangles were completed to a rectangle.

11. E = Mc².

Although it is not as directly related to space and time, one cannot conclude a series of lectures on how relativity has forced us to revise our concept of spatial and temporal relations, without saying something about $E = Mc^2$, possibly the most famous equation of all time.[1,2] To understand these we will have to examine a third quantity, momentum (P).[3]

We begin by examining the nature of mass, momentum, and energy in nonrelativistic physics. The nonrelativistic behavior of these quantities is valid to a high degree of precision when all relevant speeds are small compared with the speed of light (although even then there are interesting ways to reinterpret what is going on.) The forms of and relations between nonrelativistic mass, momentum, and energy also provide strong hints about how these quantities ought to be generalized to apply to things moving at velocities comparable to that of light.

Aside from introducing the "equivalence of mass and energy" symbolized by $E = mc^2$, the relativistic theory establishes a beautiful symmetry between energy and momentum (quite closely related to the symmetry it establishes between time and space) that is not at all evident in the nonrelativistic theory. It provides yet another insight into why it is impossible to accelerate an object to speeds greater than light, and it is, of course, of enormous practical importance in designing machines that accelerate elementary particles to speeds close to that of light.

Nonrelativistic mass.

We begin by examining how mass is defined nonrelativistically[4] This turns out to be very important because the nonrelativistic definition of mass survives intact in the relativistic case too, provided one adds a small footnote.[5]

First consider two ways *not* to define mass.

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[1] Except, perhaps, for $c^2 = a^2 + b^2$, which we have also been able to put to good use.

[2] Because these notes are somewhat heavy with equations, I have tried to summarize the argument with a bare minimum of equations in Appendix C at the end. The proper strategy is to read the main notes first, and then read Appendix C to check that you understood everything. But you might also find it helpful to glance at Appendix C at the start to get an overall picture of where we are heading.

[3] I have no idea why momentum is always denoted by P or p, except that it has to be something other than M or m, which are reserved for mass.

[4] I remind you that "non-relativistically" does not mean "ignoring the principle of relativity"; on the contrary, it means maintaining the principle of relativity, but examining how it operates when all speeds are small compared with the speed of light c.

[5] Which we shall do at the appropriate moment.

1. *First bad definition of mass (Newton).* Newton defined mass to be "quantity of matter." This is useless for two reasons:

(a) How do you count up the quantity of matter in something? If all matter was built out of identical little bricks one might be able to do it by counting the number of bricks, but unfortunately matter (as we understand it today) is made up of many different kinds of bricks, so this doesn't work unless you have an independent definition for the "quantity of matter" in two bricks of different types.

(b) Furthermore even when you put together identical bricks it turns out, as we shall see, that the mass of the resulting object can depend on how you stick the bricks together. The stronger they are bound to each other, the more the mass of the composite object falls short of the sums of the masses of the bricks that make it up.

2. *Second bad definition: Mass is weight.* The problems with this definition form a tedious but essential part of all introductory physics courses. The problem is that the weight of an object is the force that gravity exerts on it. This depends on where the object is. An object's weight on the moon is about a sixth of its weight on earth (which itself varies a bit depending on where you are on the earth) and its weight far away from other objects in empty space — in some sense the most natural environment of all — is zero.

Although Newton gave us the first bad definition, he also taught us what we need to know to construct the correct definition (and, I like to think, would have readily agreed that this was a better way to put it): *Correct definition of mass.* The mass of an object is a measure of how hard it resists attempts to change its velocity. Under a given set of circumstances, the bigger the mass of an object, the less its velocity changes.

This is far too informal a statement to stand by itself, but it captures the essential quality of the concept of mass. It's easy to push around a beachball, harder to push around a solid wooden ball of the same size, and extremely hard to push around a solid lead ball of that size. Defining mass in terms of velocities is also well suited for reexamining the concept in the relativistic case, since under such a definition one can reduce the measurement of a mass to the measurement of certain times and distances in an appropriately designed experiment.

But to make the concept of mass more precise we must go beyond this informal definition and state some simple facts, which make it possible to give a quite precise definition of mass. We do this by returning to the kinds of collisions we considered in Lecture Notes #1, in which two particles come together with certain velocities, collide, and go off with certain other velocities. We must now take care to distinguish explicitly between velocity and speed. I shall follow the widespread practice of using bold face letters ($\mathbf{u}$) for velocities, which can be positive or negative, and italic letters (u) for speeds, which are always positive. Thus a particle with speed u has velocity $\mathbf{u} = u$ if it moves to the right, and $\mathbf{u} = -u$ if it moves to the left. Note that the square of a velocity is the same as the square of the corresponding speed: $u^2 = \mathbf{u}^2$. The correct and precise definition of mass is contained in the following crucial fact about collisions between particles:

It is possible to associate with every particle a positive number m, called its mass, which is a measure of how little the velocity of the particle changes in a collision: the bigger the mass, the smaller the change in velocity. To give the precise relation between the masses of two colliding particles and their changes in velocity, call the particles 1 and 2, and their masses m_1 and m_2. Call their velocities before the collision $\mathbf{u}_1^b$ and $\mathbf{u}_2^b$, and their velocities after the collision $\mathbf{u}_1^a$ and $\mathbf{u}_2^a$, so that the *changes* in their velocities are $\mathbf{u}_1^a - \mathbf{u}_1^b$ and $\mathbf{u}_2^a - \mathbf{u}_2^b$. The results of many experiments, taken together, establish the useful fact that the comparative size of the two changes in velocity is entirely determined by the comparative size of the two masses, according to the simple rule:

$$\frac{\mathbf{u}_1^a - \mathbf{u}_1^b}{\mathbf{u}_2^a - \mathbf{u}_2^b} = -\frac{m_2}{m_1}. \tag{11.1}$$

Several comments:

1. Since the masses are positive, the minus sign simply means that the ratio of the changes in velocities is negative— i.e. the change in the velocity of one of the particles is in the opposite direction from the change in velocity of the other. If the velocity of one increases, the velocity of the other decreases.[6]

2. In the nonrelativistic theory this rule holds whatever the individual velocities happen to be. One might expect there to be trouble when the speeds approach relativistic values (significant fractions of a foot per nanosecond), and indeed the rule then fails to hold, as we shall see. However even in the more accurate relativistic theory, as one also might expect — indeed, as one ought to require — the rule holds to a very high degree of precision provided all particle speeds are small compared with the speed of light. This makes it possible to use the nonrelativistic definition of mass emerging from (11.1) to define mass even in the relativistic theory of energy and momentum. One simply makes one additional proviso: all the particle speeds in a collision that is designed to compare the masses of two particles must be small compared with the speed of light.[7]

3. Implicit in the definition (11.1) of mass is the very important experimental fact that exactly the same number m works for a given particle *regardless of what other particle it*

[6] Note that increasing or decreasing *velocity* is not the same as increasing or decreasing *speed*. If a particle moving to the right slows down a little its velocity decreases. But if a particle moving to the left speeds up its velocity also decreases, because it becomes a larger negative number. And if a particle moving to the left slows down a little its velocity increases, because it becomes a smaller negative number.

[7] "How small?" you might ask. That depends on how accurately you want to know the ratio of the masses. Since no mass is known to better than about ten significant figures, an error of one part in ten billion is good enough for all practical purposes which, as we shall see, means the speeds ought to be less than a hundred thousandth of the speed of light, or about 10 feet per millisecond — roughly 10 times the speed of sound in air — still a pretty brisk clip.

collides with. So although our definition gives only the comparative resistance to changes of velocity of a pair of particles, we end up with the same collection of masses for all the particles regardless of which pairs we choose to test against each other. Thus by testing 1 and 2 we learn the ratio m_2/m_1 and by testing 2 and 3 we learn m_3/m_2. The product of these two ratios is m_3/m_1 and indeed, if we test 1 and 3 directly this is precisely what we get. There is nothing in the nature of collision experiments that logically requires that this should be so. It is a very important fact about the world that different kinds of particles behave in this very simple way when they collide at nonrelativistic speeds. Of course we can only determine in this way the *ratio* of the masses of all the particles. The overall scale is arbitrary, and can be fixed, for example, by taking one standard object and declaring its mass to be "one kilogram".

4. Note that this definition of mass is consistent with the principle of relativity. The numbers you get for the mass ratios do not depend on the frame of reference in which the collision is described, *provided* we use the *nonrelativistic* velocity addition law. For if we view all the collisions in a frame moving to the right with speed v—i.e. with a velocity[8] $\mathbf{v}$ that is positive—then every velocity $\mathbf{u}$ appearing in (11.1) is replaced by $\mathbf{u} - \mathbf{v}$, which leaves *changes* in velocity, which are all that appear in (11.1), unaffected. This provides a strong indication that something goes awry in the relativistic case, for the relativistic rule is that when you change frames of reference $\mathbf{u}$ is replaced by $\frac{\mathbf{u}-\mathbf{v}}{1-\mathbf{u}\mathbf{v}/c^2}$, and as a result the ratio of masses deduced by examining the same collision in two different frames will no longer be the same. Of course if both speeds u and v are small compared with c this difference is so small as to be unimportant, which is why the nonrelativistic definition of mass remains valid in the relativistic theory but with the proviso that all speeds should be small compared with that of light. That the facts about the world implicit in the definition of mass should be valid in any inertial frame of reference is, of course, crucial, for the principle of relativity requires such facts to be valid in all inertial frames of reference.

Nonrelativistic momentum.

With a little elementary algebraic manipulation we can rewrite (11.1) in the mathematically equivalent form:

$$m_1\mathbf{u}_1^b + m_2\mathbf{u}_2^b = m_1\mathbf{u}_1^a + m_2\mathbf{u}_2^a. \tag{11.2}$$

Although this has precisely the same mathematical content as (11.1) it presents the information in a somewhat different way, for the left side of (11.2) only contains velocities

[8] Do not confuse the velocity $\mathbf{v}$ of the new frame of reference with the velocities $\mathbf{u}$ of the particles participating in the collision: $\mathbf{v}$ is fixed throughout the collision and has nothing to do with the collision itself. It is merely the relative velocity of the two frames whose descriptions of the collision we are interested in comparing. The individual particle velocities $\mathbf{u}$, on the other hand, can vary from one particle to another and can change in the course of the collision.

before the collision, while the right side only contains velocities after. We have therefore discovered a quantity that is unchanged, or "conserved", by the collision. It is called the total momentum, usually denoted by the symbol $\mathbf{P}$. We call it "total" momentum because it is convenient also to define the momentum $\mathbf{p}$ of an individual particle of mass m and velocity u by

$$\mathbf{p} = m\mathbf{u}, \tag{11.3}$$

so that the total momentum $\mathbf{P}$ of two particles is just

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2. \tag{11.4}$$

Eq. (11.2) is called the "Law of Conservation of Momentum". From our point of view it is just a reformulation of our definition of mass. But like that "definition" it has profound physical content going well beyond a merely conventional definition. It is a remarkable *fact* that it is *possible* to assign to every particle a number m in such a way that momentum is indeed conserved in all collisions between all possible pairs of particles.

Conservation of momentum continues to hold under conditions even more general than those I have just described. Not surprisingly,[9] it continues to hold when more than two particles participate in the collision. It also continues to hold even when the motion of the particles is not confined to a single line. In that case one must specify the velocity of a particle by its components along three different directions (for example, up–down velocity, north–south velocity, and east–west velocity). The generalized law then says that momentum is independently conserved for each of these three different components.

Rather more surprisingly, conservation of momentum continues to hold even when the numbers or kinds of particles *change* as a result of the collision. Suppose, for example, particles 1 and 2 stick together to form a single new particle, particle 3. When this happens the mass of particle 3 turns out to be just the sum of the masses of the original two,[10] and momentum continues to be conserved. Note, in this case, that it is absolutely crucial that m_3 should be $m_1 + m_2$. If it were not, then momentum could not be conserved in all frames of reference. For if all the velocities $\mathbf{u}$ are replaced by $\mathbf{u} - \mathbf{v}$, then the momentum before the collision is reduced by $(m_1 + m_2)\mathbf{v}$, while the momentum after the collision is reduced by $m_3\mathbf{v}$. Thus if m_3 were not $m_1 + m_2$, momentum would not be conserved in the new frame. This is so important that it is stated as a Law of Conservation of Mass:[11] if two particles m_1 and m_2 merge into a single particle of mass m_3, then

$$m_3 = m_1 + m_2. \tag{11.5}$$

[9] It's not surprising if you view collisions involving more than two particles as consisting of a sequence of collisions between different pairs, in the limit as the time between the different pair collisions becomes extremely short.

[10] We shall see that this is only true (but to a very high degree of accuracy) when the speeds of the particles are small compared with that of light.

[11] Our goal, $E = mc^2$, is related to the fact that this law too must often fail in the relativistic case, as we shall see.

If the Law of Conservation of Mass did not hold, then the Law of Conservation of Momentum could not hold either.

Nonrelativistic energy.

We started off interested not only in mass M, but also in energy, E. To see how E enters the picture, it is very useful to examine a two–particle collision in a very special frame of reference, in which the total momentum is zero. In this zero-momentum frame[12] we have before the collision

$$m_1\mathbf{u}_1^b + m_2\mathbf{u}_2^b = 0 \tag{11.6}$$

and, because momentum is conserved,

$$m_1\mathbf{u}_1^a + m_2\mathbf{u}_2^a = 0 \tag{11.7}$$

after the collision too. In the zero-momentum frame the particles move in opposite directions, since the velocities of 1 and 2 have to have opposite signs if their momenta add up to give zero. So in the zero-momentum frame the particles come together and then fly apart with speeds whose ratios are the same both before, and after the collision:

$$\frac{u_2^b}{u_1^b} = \frac{m_1}{m_2} = \frac{u_2^a}{u_1^a}. \tag{11.8}$$

But although the *ratios* of the speeds are the same both before and after the collision, there is nothing in the law of conservation of momentum to require the speeds *individually* to stay the same. There is, however, something special about a collision in which the speeds themselves remain the same— i.e. in which the particles simply bounce back in the directions they came from with their original speeds. One calls such collisions *elastic*, and calls *inelastic* those collisions in which the individual speeds change. Since the ratios of the speeds are the same before and after, if a collision is inelastic then both speeds are either reduced or increased. An inelastic collision in which the speeds dropped might be one in which the particles tended to stick together when in contact, and therefore lost some of their speed in the course of pulling apart again. An inelastic collision in which the speeds increased might be one in which a small explosive charge was set off when the particles touched, propelling them back faster than they came together.[13]

Whatever the reason for a collision being elastic or inelastic, however, one singles out elastic collisions for special treatment, because in an elastic collision something else, besides momentum, is conserved. In the zero-momentum frame of two particles, it is the individual

[12] A term preferred by physicists is "center of mass frame", but we shall use the more directly descriptive name.

[13] It is an important fact that momentum continues to be conserved even in cases like these.

speeds themselves that are conserved, but that is special to both the zero-momentum frame and the case of two-particle collisions. It is, however, easy to see what the new quantity must be if we want it to be conserved in *all* frames of reference. Define the "kinetic energy" k of a particle of mass m and velocity $\mathbf{u}$ by[14]

$$k = \tfrac{1}{2}m\mathbf{u}^2, \tag{11.9}$$

and define the total kinetic energy of two particles to be

$$K = k_1 + k_2. \tag{11.10}$$

Since u_1 and u_2 are *separately* conserved in an elastic collision in the zero-momentum frame, so are k_1 and k_2 and hence their sum. Any number of other possible definitions of K would share this simple property. What makes the particular definition (11.9) special is that if K is conserved in *one* frame of reference it will necessarily be conserved in *all* frames. Once we have established this we no longer need to use the zero-momentum frame to check on whether or not a collision is elastic. We need only compute $K = \frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2$ in whatever frame suits our convenience, both before and after the collision. The collision is elastic if and only if K is the same before and after.

So how does K change when we change frames? The velocity $\mathbf{u}$ changes to $\mathbf{u} - \mathbf{v}$, so the kinetic energy $k = \frac{1}{2}m\mathbf{u}^2$ changes to

$$k' = \tfrac{1}{2}m(\mathbf{u} - \mathbf{v})^2 = \tfrac{1}{2}m\mathbf{u}^2 - m\mathbf{u}\mathbf{v} + \tfrac{1}{2}mv^2 = k - \mathbf{p}\mathbf{v} + \tfrac{1}{2}mv^2. \tag{11.11}$$

If we have two particles, we just add up the changes in kinetic energy for each of them, so the total kinetic energy in the new frame is

$$K' = K - \mathbf{P}\mathbf{v} + \tfrac{1}{2}M\mathbf{v}^2, \tag{11.12}$$

where $\mathbf{P}$ is the total momentum and M is the total mass. Suppose the kinetic energy in the original frame K is the same before and after the collision. Then since the total momentum $\mathbf{P}$ and the total mass M are also unchanged by the collision, it follows from (11.12) that the kinetic energy K' in the new frame must be the same before and after the collision, since it only depends on K, $\mathbf{P}$, and M.

Thus it is a consequence of the conservation of total momentum and total mass, that if total kinetic energy is conserved in one frame, it will be conserved in all frames. If we define a collision to be elastic if kinetic energy is conserved in the collision, then the distinction between an elastic and an inelastic collision is independent of the frame of reference in which the kinetic energy has been calculated.

[14] The factor $\frac{1}{2}$ is entirely a matter of convention, designed to make things come out simpler further on. Clearly we could redefine any of these quantities (m, p, or k) by introducing arbitrary numerical scale factors that were the same for all particles.

Summary of the nonrelativistic conservation laws

Here is a summary of the nonrelativistic state of affairs:

Mass. We associate with each particle a mass m which is a number characteristic of the particle, independent of the frame of reference in which the particle is described; the total mass M of a collection of particles is just the sum of their individual masses. Total mass is conserved in all collisions. Total mass is also the same in all frames of reference. Putting this last remark more formally, if M is the mass in one frame and M' is the total mass in a frame moving with velocity $\mathbf{v}$, then

$$M' = M. \tag{11.13}$$

Momentum. If a particle of mass m has a velocity $\mathbf{u}$ we define its momentum $\mathbf{p}$ by

$$\mathbf{p} \ = m\mathbf{u}. \tag{11.14}$$

The total momentum $\mathbf{P}$ of a collection of particles is just the sum of their individual momenta. The total momentum is conserved in all collisions. The momentum $\mathbf{P'}$ in a frame moving with velocity $\mathbf{v}$ is related to the momentum $\mathbf{P}$ in the original frame by

$$\mathbf{P'} = \mathbf{P} - M\mathbf{v}. \tag{11.15}$$

where M is the total mass.

Energy. If a particle of mass m has a velocity $\mathbf{u}$ we define its kinetic energy k by

$$k = \tfrac{1}{2}mu^2. \tag{11.16}$$

The total kinetic energy K of a collection of particles is just the sum of their individual kinetic energies. The total kinetic energy is only conserved in a special kind of collision, known as an elastic collision. The kinetic energy K' in a frame moving with velocity $\mathbf{v}$ is related to the kinetic energy K in the original frame by

$$K' = K - \mathbf{P}\mathbf{v} + \tfrac{1}{2}Mv^2. \tag{11.17}$$

where M is the total mass and $\mathbf{P}$ is the total momentum in the original frame.

Note here the interplay between conservation laws (quantities which are the same before and after the collision) and transformation rules (which tell how quantities change from one frame of reference to another). A conservation law relates the value of a quantity before the collision to its value after the collision, when both values are computed in the same frame of reference. For it to be a *law* it must be valid in all frames of reference, so we must use the transformation laws to check that a candidate for a conservation law is capable of being obeyed in all frames of reference. In the case of mass conservation that is easy, since mass is the same in all frames of reference. Momentum can be conserved in all frames of reference because it obeys the transformation rule (11.15) *and* because the total mass is the same before and after a collision. Kinetic energy can be conserved in all frames of reference (if it is conserved in any one frame) because it obeys the transformation rule (11.17) *and* because *both* the total momentum *and* the total mass are the same before and after a collision.

Note also the important fact that the *contingently* conserved quantity, K, does not appear in the transformation rules governing the quantities $\mathbf{P}$ and M that are *always* conserved. If K did appear in the transformation rules for either $\mathbf{P}$ (or M), then since K is not always conserved, neither could $\mathbf{P}$ (or M) always be conserved.

Relativistic mass, momentum, and energy.

When we get to speeds comparable to the speed of light, this simple nonrelativistic picture falls apart. As already noted, the pleasing compatibility of these conservation laws and their ability to be satisfied in all frames of reference makes critical use of the nonrelativistic velocity addition law, $\mathbf{u}' = \mathbf{u} - \mathbf{v}$. When this rule is significantly violated, then conservation of momentum ceases to be a rule that holds in all frames if it holds in any one, because the simple transformation rule (11.15) for momentum is no longer valid. The same problem arises with kinetic energy. This is not surprising. There is no reason to expect that the appropriate forms for the momentum and kinetic energy of a high speed particle should be identical to the forms they have at nonrelativistic speeds. After all, not even the rate of a moving clock or the length of a moving stick is the same as it is in the nonrelativistic case. The question we must address is whether it is possible to find new conservation laws involving suitable generalizations of the nonrelativistic definitions of mass, momentum, and kinetic energy.

These generalizations must have two crucial features: (a) They must reduce to the nonrelativistic forms when the speeds of the particles are small compared with the speed of light, since we know the nonrelativistic conservation laws hold to a high degree of accuracy in that limit; (b) If the appropriately generalized quantities are conserved in one frame of reference then they must be conserved in all frames of reference, or we could distinguish between different inertial frames of reference by doing an experiment to see whether, for example, momentum was or was not conserved.

The proper relativistic definition of mass is the easiest to deal with. As remarked upon above, we retain exactly the same definition of mass as in the nonrelativistic theory, only adding the proviso that the velocities of all particles in a collision used to determine their masses should be small compared with the velocity of light.[15,16] As so defined, the mass of a particle continues to be an inherent property of the particle, having nothing to do with how fast the particle might be moving in collisions it might subsequently find itself in. It is an invariant, independent of frame of reference.[17]

We defer for the moment the question of whether or not total mass, defined as the sum of the masses of the individual particles, continues to be conserved in collisions that change the numbers and types of particles. Note, though, that any failure of mass conservation had better be by a very small amount when the speeds of all particles participating in the collision are small compared with the speed of light, since the nonrelavistic theory, in which total mass *is* conserved, holds to a high degree of precision when all speeds are small compared with c.

We turn next to the relativistic definition of the momentum of a particle of mass m. Since m continues to be simply an invariant number, characterizing the particle, we must decide what quantity can play the role of the particle's velocity in generalizing the non-relativistic definition $p = m\mathbf{u}$. We have two criteria to meet: (a) the new quantity must reduce to $\mathbf{u}$ when u is small compared with c; (b) when one changes frames of reference the new quantity must change in a manner that has a simplicity comparable to the nonrelativistic rule $\mathbf{u}' = \mathbf{u} - \mathbf{v}$, if we are to have any hope of conserving momentum in all frames of reference. The velocity $\mathbf{u}$ itself will not do, for under a change of frame of reference $\mathbf{u}$ changes by the relativistic law:

$$\mathbf{u}' = \frac{\mathbf{u} - \mathbf{v}}{1 - \mathbf{uv}/c^2}. \tag{11.18}$$

It is the denominator in (11.18) that prevents the transformed total momentum $\mathbf{P}' = m_1\mathbf{u}_1' + m_2\mathbf{u}_2'$ having a form simple enough to ensure momentum conservation in the new frame. The problem is that if we continue to define momentum by (11.14) but use the relativistic transformation law (11.18), then the total momentum in the new frame of reference depends in detail on the individual velocities of the particles in the old frame, instead of depending on those velocities only through that particular combination of velocities which is nothing but the total momentum in the old frame, as is the case in the non-relativistic relation (11.15).

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[15] How small, as noted, depends on how accurately we want to determine the masses. A good practical criterion is to say that they should be so small that if we repeat the experiment with even smaller velocities, we get exactly the same set of masses to within the accuracy of the method we use to determine the relevant speeds.

[16] But. you may protest, that does us no good in determining the mass of a photon, since photons in empty space cannot move at any speed other than the speed of light. I return to the special case of photons at the end of this chapter.

[17] If there were a particle whose mass was not invariant, then we could distinguish one inertial frame from another by performing in each frame a low velocity collision that determined the mass of the particle.

Why does relativity introduce a complicated denominator into (11.18)? Think back to the definition of velocity: distance travelled divided by the time it takes. In the nonrelativistic case changing frames changes the distance traveled, *but does not change the time it takes,* so only the numerator changes. In the relativistic case *both* quantities change when you change frames, leading to the more elaborate rule (11.18). This suggests an extremely simple and ingenious way out of the problem:

Suppose we had a generalization of the velocity of a particle which was the distance it traveled divided by some time that did *not* depend on frame of reference. The hope would be that if we measured the distance travelled in the usual frame-dependent way, but measured the time that it took the particle to go that distance in a special frame that all observers could agree on, then this generalized particle velocity might change sufficiently simply under a change of frames to make it possible to resurrect momentum conservation in the relativistic case.

But what could such a special frame be? To ask the question is to answer it. The situation singles out one and only one special frame—*the frame of reference moving with the particle itself.*

Let us define a generalized velocity $\mathbf{w}$ to be the distance a particle travels in a given time with the proviso that *this time should always be measured by a clock travelling with the particle.* Note at once that $\mathbf{w}$ reduces back to the ordinary velocity $\mathbf{u}$ when the speed of the particle is small compared with the speed of light, since a clock moving with the particle then runs slowly by an imperceptibly small amount. Now, however, as we go from one frame to another only the distance going into the definition of $\mathbf{w}$ changes, but not the time. So if we redefine the momentum $\mathbf{p}$ to be $m\mathbf{w}$, we might hope to find a simple transformation rule for $\mathbf{p}$. Since m is invariant, we must inquire how $\mathbf{w}$ transforms.

Now $\mathbf{w}$ is defined in exactly the same way as $\mathbf{u}$ except that the motion of the particle is timed by a clock that is not stationary, but moving with the particle. Such a clock runs *slowly*, so compared with stationary clocks it will indicate that the particle took *less* time to cover a given distance. The reduction in time is precisely by the slowing-down factor $s = \sqrt{1 - u^2/c^2}$, and therefore $\mathbf{w}$ is *bigger*[18] than the ordinary velocity $\mathbf{u}$ by precisely the factor $1/\sqrt{1 - u^2/c^2}$:

$$\mathbf{w} = \mathbf{u}/\sqrt{1 - u^2/c^2}. \tag{11.19}$$

With the definition (11.19) at hand, we can use the transformation rule (11.18) to find how $\mathbf{w}$ changes when we change to frame moving with velocity $\mathbf{v}$. In the new frame $\mathbf{w}'$ is given by

$$\mathbf{w}' = \mathbf{u}'/\sqrt{1 - u'^2/c^2}, \tag{11.20}$$

[18] It is bigger because the particle can go a greater distance in one second when that second is measured by its own clock, than when the second is measured by the clocks in the frame in which we are describing the particle's motion, since the particle's clock runs slowly compared with our clocks.

where $\mathbf{u}'$ is related to $\mathbf{u}$ by the velocity addition law (11.18). If you substitute (11.18) into (11.20) and simplify the resulting expression[19] you will find that

$$\mathbf{w}' = \frac{\mathbf{u} - \mathbf{v}}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}}. \tag{11.21}$$

If we now define relativistic momentum by

$$\mathbf{p} = m\mathbf{w} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}, \tag{11.22}$$

then (11.21) tells us that

$$\mathbf{p}' = \frac{\mathbf{p} - p_0\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \tag{11.23}$$

where I have defined a new quantity p_0 by

$$p_0 = \frac{m}{\sqrt{1 - u^2/c^2}}. \tag{11.24}$$

This is close to what we want, for the momentum in the new frame is now very simply[20] related to the momentum in the old frame. The only problem is that something new has appeared as well, p_0. To see what this might signify, let us first consider what happens when the speed u of the particle is small compared with the speed of light. In that case (11.24) tells us that p_0 is indistinguishably different from the mass m. If we replaced p_0 by m in the transformation law (11.23), and applied it to the total momentum of a pair of particles, we would get

$$\mathbf{P}' = \frac{\mathbf{P} - M\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \tag{11.25}$$

which except for the denominator is just the familiar nonrelativistic transformation law. The denominator is harmless, however, since it is just a fixed number, that remains the

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[19] Checking this result is the only slightly messy piece of algebra in this whole business, but the conclusions it leads to are so profound that everybody should suffer through it at least once in a lifetime. If you have not the stomach to do the algebra, at least note that (21) is obviously correct when $\mathbf{v} = 0$ (in which case it reduces to $\mathbf{w}' = \mathbf{w}$), when $\mathbf{v} = \mathbf{u}$ (in which case we have gone to a frame in which the velocity of the particle is 0), and when $\mathbf{u} = 0$ (in which case the particle was originally at rest and therefore has velocity $-\mathbf{v}$ in the new frame.)

[20] The factor $\sqrt{1 - v^2/c^2}$ in the denominator in (11.23) may not strike you as so simple, but remember that it is only a number, determined by the relative velocity of the two frames. It is independent of the speed of the particle itself, and therefore exactly the same number will appear in the rule giving the momentum in the new frame of every single particle that participates in the collision.

same before and after the collision,[21] and we can conclude from (11.25) just as we did in the nonrelativistic case, that if $\mathbf{P}$ is the same before and after a collision then $\mathbf{P}'$ will be too, provided the total mass M is conserved in the collision.

But if a particle is not moving at a speed small compared with c then p_0 is *not* its mass m. If we want to insure that momentum, as defined by (11.22) is conserved in all frames of reference,then we must then replace the law of conservation of total mass by a new law of conservation of total p_0. Such a replacement is in keeping with the spirit of our attempted generalization of the nonrelativistic conservation laws, for total p_0 is given by

$$P_0 = p_0^1 + p_0^2 = \frac{m_1}{\sqrt{1 - \mathbf{u}_1^2/c^2}} + \frac{m_2}{\sqrt{1 - \mathbf{u}_2^2/c^2}}. \tag{11.26}$$

Since this reduces to total mass when both velocities are small compared with c, we are entertaining the possibility that the nonrelativistic law of mass conservation is also a limiting case of a more general relativistic law, just as the nonrelativistic law of conservation of total $m\mathbf{u}$ is a limiting case of conservation of a more general relativistic concept of momentum.

But before we can declare there to be a new conservation law for P_0, we must check to see whether it too passes the crucial requirement that a genuine law must hold in all frames of reference. This leads us to one more unpleasant computation very much like to the one that led us to (11.23).[22] We must apply (11.18) to the definition

$$p_0' = \frac{m}{\sqrt{1 - u'^2/c^2}} \tag{11.27}$$

to express p_0' in terms of quantities in the original frame. When this is done we find:

$$p_0' = \frac{p_0 - \mathbf{p}\mathbf{v}/c^2}{\sqrt{1 - v^2/c^2}}. \tag{11.28}$$

This has a structure very similar to the transformation rule (11.23) for momentum. Because both structures are so simple, the transformations (11.23) and (11.28) for the individual particle $\mathbf{p}$ and p_0 lead to transformations for the total momentum $\mathbf{P}$ and total P_0 of exactly the same forms as (11.23) and (11.28) :

$$\mathbf{P}' = \frac{\mathbf{P} - P_0\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \tag{11.29}$$

[21] Remember that v is just the relative speed of the two frames, and not the speed of any of the particles.

[22] You can extract the result (28) more deftly by dividing the left side of the momentum transformation law (11.23) by the left side of the velocity transformation law (11.18) and the right side by the right side, and comparing what you get with the definitions of p_0 and $\mathbf{p}$.

$$P_0' = \frac{P_0 - \mathbf{P}\mathbf{v}/c^2}{\sqrt{1 - v^2/c^2}}. \tag{11.30}$$

Since these express $\mathbf{P}'$ and P_0' entirely in terms of $\mathbf{P}$ and P_0 (and the relative velocity $\mathbf{v}$ of the two frames) *if the unprimed quantities are the same before and after a collision, the primed quantities must be too.* Therefore if $\mathbf{P}$ and P_0 are both conserved in one frame they will both be conserved in any other frame. Our proposed relativistic generalization (11.22) of the definition of momentum meets all of our criteria for a conserved quantity, as does the new quantity P_0 whose conservation we are considering.

What are the implications of replacing the nonrelativistic conservation of total mass M by the relativistic conservation of P_0? How are we to interpret p_0 and the sum P_0 of the values of p_0 for a group of particles? We can get a powerful clue by examining the structure of p_0 for a particle whose speed u is small compared with c. In this limit the definition (11.24) merely tells us what we already know: that p_0 is very close to m, the mass of the particle. But since we are trying to make sense of the *difference* between the old nonrelativistic law of conservation of M and a new relativistic law of conservation of P_0, what we really require is an estimate of the difference between p_0 and m when u is small compared with c that does better than a simple declaration that the difference is very small. In the Appendix A at the end of these notes we construct such an estimate,[23] showing that when u is very small compared to c, then to an exceedingly high degree of accuracy,

$$p_0 - m = \tfrac{1}{2}mu^2/c^2. \tag{11.31}$$

Thus at nonrelativistic velocities $p_0 - m$ *is nothing but the nonrelativistic kinetic energy divided by* c^2.

So if we define the *relativistic kinetic energy* by

$$k = p_0c^2 - mc^2, \tag{11.32}$$

then k does indeed reduce to the ordinary nonrelativistic kinetic energy at speeds small compared to c and we have our interpretation of p_0: the interesting quantity is not p_0 itself, but the product of p_0 with c^2, which (11.31) tells us is the sum of two terms:

$$p_0c^2 = mc^2 + k. \tag{11.33}$$

We have almost reached our goal. In order for the relativistic momentum $\mathbf{P}$ to be conserved it is necessary for P_0 to be conserved as well. But

$$P_0c^2 = Mc^2 + K, \tag{11.34}$$

[23] The analysis is too simple to require an Appendix, but I can't stand to interrupt the narrative at this exciting moment.

where M is the total mass and K, the total kinetic energy.

Recall now the nonrelativistic state of affairs. Total mass M is always conserved, but total kinetic energy K is only conserved in elastic collisions. Relativistically we can continue to define elastic collisions as those in which K is conserved. But relativistically P_0 must *always* be conserved.[24] Since P_0 is simply related to M and K, it follows from (11.34) that if K is conserved then M must be conserved as well. *But if K is not conserved, then M cannot be conserved either.* In an inelastic collision if the total kinetic energy goes down (or up) by[25] ΔK then in order for P_0 to be conserved in the collision, (11.34) requires that the change in kinetic energy must be precisely balanced by an increase (or decrease) in the total mass by ΔM, where

$$\Delta M c^2 = \Delta K. \tag{11.35}$$

This must be true whether the collision involves relativistic or nonrelativistic velocities, since the relativistic theory ought to be valid for all velocities. Why, then did we never notice it in inelastic collisions at nonrelativistic velocities, where total mass appeared to be conserved? The reason is that the change in mass is then extremely small. The change of mass in an inelastic collision is $\Delta M = \Delta K/c^2$, and a measure of the size of ΔK, the change of kinetic energy, is the total mass M times the square of a typical particle velocity u^2. Thus the change in ΔM is typically the mass M itself times a factor whose size is roughly u^2/c^2. At less than supersonic velocities, u^2/c^2 is less than 1/1,000,000,000,000.

So the change in mass required in inelastic collisions by the relativistic theory is simply too small to be noticed in nonrelativstic collisions. The exact relativistic conservation of P_0c^2 simply masquerades as conservation of total mass. But at relativistic speeds the consequences of the correct relativistic conservation law can be profound.

Returning from the sublime to the merely conventional, I note that one defines P_0c^2 to be E, the total energy, and defines $\mathbf{p}_0c^2$ for each individual particle to be its energy e. The energy and momentum of a particle of mass m and velocity u are defined by

$$e = \frac{mc^2}{\sqrt{1-u^2/c^2}}, \tag{11.36}$$

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}. \tag{11.37}$$

The rules (11.29) and (11.30) become

$$E' = \frac{E - \mathbf{P}\mathbf{v}}{\sqrt{1-v^2/c^2}}, \tag{11.38}$$

[24] For if it were not, remember, momentum could not be conserved in all frames.

[25] By ΔK we just mean the change in K as a result of the collision: $\Delta K = K^a - K^b$, and similarly, $\Delta M = M^a - M^b$.

$$\mathbf{P}' = \frac{\mathbf{P} - E\mathbf{v}/c^2}{\sqrt{1 - v^2/c^2}}. \tag{11.39}$$

Note that (11.36) asserts that the energy e of a particle of mass m has the value mc^2 when the particle is at rest. This is sometimes incorrectly cited to be the meaning of $E = Mc^2$. But this by itself is merely an uninteresting matter of convention. One could equally well have defined the energy of a particle to be its kinetic energy, $k = e - mc^2$, in which case the energy of a particle at rest would be zero. The true meaning of $E = Mc^2$ is to be found in the study of inelastic collisions, as the expression (11.35) of the balance between changes in total kinetic energy and changes in total mass. Thus if two particles collide in their zero-momentum frame and stick together to form a final particle at rest, the mass of that final particle will exceed the sum of the masses of the two incident particles by precisely their kinetic energy prior to the collision divided by c^2. Conversely, if a particle at rest spontaneously disintegrates into two particles that go flying off, the total mass of the two particles must be less than the mass of their parent by precisely their kinetic energy divided by c^2.

If you wish to create new particles, more massive than any that have been observed to date, it is necessary to fling together less massive particles at speeds close to c to provide the kinetic energy needed to supply the additional post-collision mass. This is a matter that was of considerable interest to the Congress of the United States and the economy of the state of Texas in the early 1990's. Less expensive and still highly viable versions of the same process take place on the Cornell campus under Upper Alumni Field.[26]

Photons.

Note that (11.36) implies that the energy of a particle becomes arbitrarily large as its speed approaches that of light—yet another illustration of the difficulty of accelerating anything up to the speed of light. Yet there are particles (the photon, for example) that do move at the speed of light. How are we to account for this? Evidently (11.36) allows a particle to move at a speed u equal to the speed c of light without requiring an infinite amount of energy to do so, provided the mass of such a particle is zero. At first glance it appears that (11.36) and (11.37) can tell us nothing useful about zero mass particles with speeds $u = c$, since dividing 0 by 0 is a famous way of arriving at utter nonsense. But in fact there are two consequences of these two equations that remain perfectly well defined in the limit of zero m.

It follows from (11.36) and (11.37) that

$$e^2 = p^2c^2 + m^2c^4. \tag{11.40}$$

[26] Cornell has been for many years the last university in the United States where such experiments continue to be done under the direct management of the physics epartment.

and that

$$p = eu/c^2. \tag{11.41}$$

Indeed Eqs. (11.41) and (11.40) are completely equivalent to (11.36) and (11.37),[27] but they have the virtue of retaining an intelligible content even when applied to particles of zero mass. When $m = 0$ (11.40) reduces to

$$p = e/c. \tag{11.42}$$

This is consistent with (11.41) provided the speed u of the zero mass particle is equal to the invariant speed c. Thus the relativistic definitions of energy and momentum apply perfectly well to a particle of zero mass, where they reduce (a) to the requirement that the speed of such a particle is necessarily c, and (b) to the condition that the energy of such a particle is just c times its momentum. This turns out to be extremely useful. For an illustration, see Appendix B.

How fast does something move through time?

There is a way to view the quantity p_0 from a somewhat different perspective, which ties together the concepts of energy and momentum in a way that is simply unavailable in the nonrelativistic case. The momentum of a particle in any given frame of reference is the product of the mass of the particle with the rate at which the particle moves through *space* as measured by a clock moving with the particle. In quite the same way p_0 is the mass of the particle times the rate at which the particle moves through *time*, as measured by a clock moving with the particle.

To nonrelativistic ears this sounds crazy: how can something move through time at anything but a rate of one second per second.[28] But relativistically it makes perfect sense as yet another way to express the slowing down of a moving clock. The higher the speed of a particle (in a given frame of reference) the more rapidly the particle moves through time (as time is measured in that frame of reference) according to a clock moving with the particle (which measures time in the proper frame of the particle). Thus in a frame in which a particle moves at $\frac{3}{5}$ the speed of light, it moves through time at a rate of $\frac{5}{4}$ of a second per proper second. This is just a dramatic, upside-down, and in some deep sense more meaningful way of saying that any internal clock-like processes associated with the particle run slowly by the appropriate slowing down factor: for every second that passes

[27] That is to say, you can reverse the process. Starting with (11.41) and (11.40) you can deduce (11.36) and (11.37). It turns out that for most purposes (11.41) and (11.40) are much easier to work with than (11.36) and (11.37) so that while (11.36) and (11.37) play a fundamental role in motivating the new definitions of energy and momentum, it is (11.41) and (11.40) that capture their most important features.

[28] And indeed, non-relativistically conservation of p_0 is just conservation of mass.

on any clock moving with the particle, time in the frame in which we are describing this motion advances by 1.25 seconds.

When something speeds up its passage through space, so that it takes it less proper time to get from here to there, it also speeds up its passage through time, so that it takes it less proper time to get from now to then.

Explosions.

When an unstable heavy atomic nucleus disintegrates, the total masses of the resulting lighter nuclei add up to about to about 0.1% less than the original mass. While this is not an enormous change in mass, it is large enough to be easily measurable.

According to (11.35) this loss of mass must be balanced by the total kinetic energy of the lighter nuclei. Consider the simple case where there are just two lighter nuclei of equal mass. If the parent nucleus had mass M, then since only a thousandth of the mass has disappeared, each lighter nucleus will have mass of just about $\frac{1}{2}M$. If they each move with speed u after the disintegration, then if u is small compared with c their combined kinetic energies will be $K = \frac{1}{2}Mu^2$. For total energy to be conserved in the disintegration this must be equal to the loss of mass, $0.001Mc^2$, so

$$u^2/c^2 = 0.002, \tag{11.43}$$

which makes u/c about 0.045. This justifies our use of the non-relativistic form for the kinetic energy of the lighter nuclei.

But although the lighter nuclei are moving at speeds small compared with the speed of light, their speeds are several percent of c, which is an enormous speed by ordinary standards (One percent of the speed of light is almost 2,000 miles a second — 10,000 times the speed of sound in air.) An immense amount of energy is liberated in such a disintegration, even though only a thousandth of the total mass has disappeared.

Exactly the same thing happens in a chemical explosion, but the energy released for a given amount of mass is about a million times less. This is because the forces that are at play inside an atomic nucleus are about a million times stronger than those that are at work in chemical reactions. Since the energy of a nuclear explosion is typically a thousandth of the total mass M times c^2, that in a chemical explosion is more like a billionth of Mc^2. Just as in a nuclear explosion, that energy must be balanced by a loss of mass, but since the loss is only a part in a billion, it is quite impossible to notice.

Appendix A: A convenient form for $p_0 - m$.

It follows from the definitions of $\mathbf{p}$ and p_0 (11.22) and (11.24) that

$$p_0^2 - \mathbf{p}^2/c^2 = m^2 \tag{11.44}$$

or

$$p_0^2 - m^2 = (p_0 - m)(p_0 + m) = \mathbf{p}^2/c^2 \tag{11.45}$$

or

$$(p_0 - m) = \frac{\mathbf{p}^2}{(p_0 + m)c^2}. \tag{11.46}$$

The left side of (11.46) is what we are looking for: the difference between p_0 and m. The right side unfortunately contains p_0 again, but if we are only interested in speeds u small compared with c, then p_0 is exceedingly close to m. Consequently when u is small compared with c we can evaluate the right side of (11.46) with very high accuracy if we replace p_0 simply by m. Under these same conditions $\mathbf{p}$ is also very close to the non-relativistic value $m\mathbf{u}$. Making both these replacements on the right side of (11.46) gives us the estimate we are looking for. When a particle moves slowly compared with the speed of light, to a high degree of precision,

$$p_0 - m = \tfrac{1}{2}mu^2/c^2. \tag{11.47}$$

Appendix B:
What happens when a photon collides with a stationary particle.

As a simple illustration of how the relativistic conservation laws work in an extreme relativistic case, consider a collision between a photon (which of course moves at the extremely relativistic speed c) and an initially stationary particle of mass m_i, in which the photon is absorbed by the particle.[29] If the photon has energy ω ("omega", a popular notational choice for the energy of a photon) how fast does the particle move after it has absorbed the photon, and what is the particle's new mass m_f? (The subscripts i and f stand for "initial" and "final".)

The answers fall directly out of the conservation laws for total energy and momentum:

Energy conservation. Before the collision the photon has energy ω and the particle has energy m_ic^2 (since this is what (11.36) — or (11.40) and (11.41) together — gives for a particle with mass m_i and speed $u = 0$.) After the collision the particle has swallowed up the photon and has energy e. Conservation of energy requires:

$$\omega + m_ic^2 = e. \tag{11.48}$$

Momentum conservation. Before the collision the photon has momentum k, which (11.42) tells us is related to its energy ω by

$$k = \omega/c. \tag{11.49}$$

[29] This is a relativistic version of the collision between the two particles that stick together and form a single compound particle after the collision.

Before the collision the particle has momentum 0, since it is stationary. After the collision it has momentum p and there is no photon left. So conservation of total momentum requires the particle to have all the momentum originally possessed by the photon:

$$\omega/c = p. \tag{11.50}$$

Now if you know the energy and the momentum of an object, then you can most easily extract its velocity directly from (11.41): if something moves with speed u its energy e and momentum p are related by $p = eu/c^2$, so the ratio of its speed to the speed of light is given by

$$u/c = cp/e. \tag{11.51}$$

Using the forms (11.50) and (11.48) for p and e gives the answer:

$$u/c = \frac{1}{1 + m_i c^2/\omega}. \tag{11.52}$$

If $m_i c^2$ is large compared with the energy ω of the photon, then the speed of the particle after the collision is a small fraction of the speed of light. But when the energy ω of the photon becomes comparable to $m_i c^2$ of the particle, the speed with which the particle recoils becomes comparable to c. To get the particle moving at speeds very close to the speed of light c, the energy ω of the photon must become much larger than $m_i c^2$. Note, though, that no matter how large ω becomes, (11.52) still gives a final speed u for the particle that is less than the speed of light.

The simplest way to get at the mass m_f of the particle after it has absorbed the photon is through the relation (11.40) between the energy, momentum, and mass of a particle. Applied to the particle after it has absorbed the photon, this gives

$$(m_f c^2)^2 = e^2 - (pc)^2. \tag{11.53}$$

Using the forms (11.48) and (11.50) for e and p we learn from (11.53) that m_f satisfies

$$(m_f c^2)^2 = (\omega + m_i c^2)^2 - \omega^2 = (m_i c^2)(2\omega + m_i c^2), \tag{11.54}$$

so the mass of the particle after it has absorbed the photon has become

$$m_f = m_i \sqrt{1 + 2\omega/m_i c^2}. \tag{11.55}$$

Thus the mass m_f of the particle after it has absorbed the photon can be significantly larger than its original mass m_i, provided the energy ω of the photon is comparable to or exceeds $m_i c^2$.

Note that one can use (11.52) to reexpress the relation (11.55) between the initial and final masses in terms of the velocity u of the final particle. The result is the curious fact the ratio of the masses is given by nothing but the Doppler shift factor:

$$m_f/m_i = \sqrt{\frac{1 + u/c}{1 - u/c}}. \tag{11.56}$$

Appendix C: Summary in the form of a table.

Conservation of momentum, energy, and mass relativistically and nonrelativistically

	Non-Relativistic	**Relativistic**
MASS	$M = m_1 + m_2$	$M = m_1 + m_2$
Conserved?	*always*	*elastic collisions only*
Transformation	$M' = M$	$M' = M$
MOMENTUM	$\mathbf{P} = m_1\mathbf{u}_1 + m_2\mathbf{u}_2$	$\mathbf{P} = \frac{m_1\mathbf{u}_1}{\sqrt{1-\mathbf{u_1}^2/c^2}} + \frac{m_2\mathbf{u}_2}{\sqrt{1-\mathbf{u_2}^2/c^2}}$
Conserved?	*always*	*always*
Transformation	$\mathbf{P}' = \mathbf{P} - M\mathbf{v}$	$\mathbf{P}' = \frac{\mathbf{P}-\mathbf{v}E/c^2}{\sqrt{1-\mathbf{v}^2/c^2}}$
ENERGY	$E = \frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2$	$E = \frac{m_1 c^2}{\sqrt{1-\mathbf{u_1}^2/c^2}} + \frac{m_2 c^2}{\sqrt{1-\mathbf{u_2}^2/c^2}}$
Conserved?	*elastic collisions only*	*always*
Transformation	$E' = E - \mathbf{P}\mathbf{v} + \frac{1}{2}M\mathbf{v}^2$	$E' = \frac{E-\mathbf{v}\mathbf{P}}{\sqrt{1-\mathbf{v}^2/c^2}}$

Comments on the Table

1. "Conserved" means that the quantity is the same before and after the collision.

2. The entries under "Transformation" give with a prime ($'$) the value the quantities have in a frame moving with velocity $\mathbf{v}$ with respect to a frame in which they have values without primes.

3. The same relations hold for any number of particles. The number of particles before and after the collision need not be the same. If there is only one particle before (or

after) the "collision" then we are describing a particle that breaks up into more than one (or several particles fusing into one).

4. When the speed of a particle is small compared with the speed of light then its energy, $\frac{mc^2}{\sqrt{1-\mathbf{u}^2/c^2}}$ is very nearly equal to $mc^2 + \frac{1}{2}m\mathbf{u}^2$.

5. In any frame of reference u is equal to the distance a uniformly moving particle goes divided by the time it takes it to go that distance; $\frac{u}{\sqrt{1-\mathbf{u}^2/c^2}}$ is equal that same distance divided, now, by the time it takes the particle to advance that much distance *as measured by a clock moving with the particle.*

6. The conservation laws obey the principle of relativity: if they hold in one inertial frame then they hold in all inertial frames. (This is true for the nonrelativistic quantities only if one uses use the nonrelativistic rules for changing frames of reference.)

7. Note the different roles played by inelastic collisions in the relativistic and nonrelativistic theories. Non-relativistically mass is conserved even in inelastic collisions but kinetic energy is not; relativistically energy is conserved even in inelastic collisions but mass is not.

12. A Relativistic Tragicomedy

One Act, set in otherwise empty space

Cast of Characters:

Alice
Eve
Bob
Flo
Chorus of Relativists

Alice, surrounded by her clocks and meter sticks, is talking with Eve

Eve: Tell me, good Alice, is it truly so
That you are in a state of perfect rest?

Alice: I am, Eve, I move not. My state of rest
Is pure and absolute.

Eve: Is it then true
Your meter sticks do span a meter's length?

Alice: Not one jot more nor less I do confess,
Provided they maintain their state of rest.

Eve: Pray tell me, in an honest hour's good time
What will your clocks have measured on their dials?

Alice: Faith, Eve, an honest hour! No more, no less,
So long as they remain with me, at rest.

Eve: And does each of your clocks, regardless of
The distance 'twixt them, read the same true time
Upon their dials, all in that sweet relation
That does befit fine clocks: Synchronization?

Alice: This too is so (once more the truth you've guessed!)
Of all my clocks that, with me, are at rest.

Eve: Your rhymes improve at couplet's grace's expense.

Alice: Blank verse is not my business. Get thee hence.

Eve: A thousand pardons, Alice! I did jest
And did not mean to agitate your rest.

Alice: My rest is perfect, absolute, and true.

Eve: In that case, Alice, must it always be

That clocks and meter sticks that pass you by
With uniform velocity (called **v**)
Fail to be synchronized, slow down, and shrink
As it is written in the Einstein Rules?

Alice: Just so, good Eve, just so. You speak the truth.

Bob now floats uniformly into view, accompanied by an immense network of clocks and meter sticks.

Eve: Look you! who comes now?

Alice: That is Robert. See:
He comes at us at constant speed (say v).
Look how his clocks do fail to read the same,
Take longer than a second to describe
A seconds passage, while his meter sticks
Do shrink in the direction of his motion,
All in accordance with my lovely rules.*

Eve: Welcome most hearty, Bob, to Alice's home.

Bob: Nay, warmest welcome to both you and Alice
As you progress toward my most proper place.

Eve: How fare your many clocks and meter sticks?

Bob: Now and fore'er, Eve, they are just and true.
My clocks are in harmonious accord
And in a second's time do indicate
The passage of a perfect passing second.
My meter sticks extend one meter's length
From end to end.

Eve: Hear you that, Alice?

Alice: I do.
The man has lost his wits. He does not know
That it is he who moves, while I stand still.
Ergo the knave is fully unaware
That all his clocks and meter sticks behave
As it is written in the Einstein rules,
Failing to keep true time and span true length
To that extent precise and mathematick

* Which in this play are called "the Einstein rules".

As do the rules require for one who moves
Past me with his velocity.

Eve: Poor fool!
But now he passes by you and will see
By observation and comparison
Of his askew equipment with yours true,
That his is deep in error.

Alice: No, alack!
You overestimate the wisdom of
The man. So deep has he enmeshed himself
In folly, so fully does he deem himself
At rest, that he believes that my own Ein-
Stein's rules describe the sticks and clocks at rest
With me!

Eve: A double folly's double woe!
But yet methinks there consolation be
In doubleness. The saving point is this:
If to his false-deemed state of rest erroneous
He adds a further concept incorrect
And vile, by his most wrongful application
Of your own Einstein's Rules, which we both know
Describe the strange distortions of things moving
Past her who is at rest (and such are you)...
If, as I say (for I have lost the thread
Of my intent) he wrongfully applies
Your special rules, assuming they are his,
Then marry, by this double error gross
(Wrongly to deem himself at rest, and worse,
Wrongly to think that he can use your rules)
Does he not double chance of contradiction
Which will his fault correct, his mind inform,
When he observes your instruments of measure
So just and true, due to their state of rest?

Alice: His second folly does abet his first
And by compounding, save it. Had he thought
Himself at rest and not as well believed
My own dear Einstein's rules, his too to use,
His error, by th' impending confrontation
Of swift advancing Bob and my true tools

Of space and time, would manifest become
To Bob himself, forced to this recognition
By contradiction stark and merciless.
Howe'er because he uses my own rules
As if 'twere he at rest and I who moved,
Along with my true clocks and meter sticks,
The inconsistencies that should inform
His intellect of its sad misconception
And jar it like a ringing clarion call
To certain knowledge of those clear distortions
His many clocks and meter sticks are heir to
By virtue of their motion, he, poor fool,
Is able to account for in a way
That masks the inconsistencies and bars
Sweet ministering congtradiction from
The portals of his mind. He simply blames
The facts that should destroy his sleep dogmatic
On the fictitious shrinkage, slowing down,
And lack of that sweet quality we deem
Most excellent in clocks, Synchronization,
That he in his most vain, deluding use
Of my own Einstein's rules assigns to my
Most wrongfully maligned instruments.
To his misfortune, Nature, arch deceiver,
So made the world that his delusions, two,
Will learn from this encounter nothing new.
Each does confirm the other's false surmise.
So was it e'en with Charles and so with Di.
So shall it be when Flo and George come by.
I rage against such cruel deceit in vain;
Harsh Nature has decreed it.

Eve: (*to Bob, now very close*) Look you, Bob:
The clocks and meter sticks of outraged Alice!
Perceive: They argue not the same as yours!

Bob: Of course they don't: Her meter sticks do shrink,
Her clocks are slow, nor are they synchronized;
While my sticks measure distance absolute,
My clocks record Time's true and even tread,
Each, though apart, my other clocks do prove.
That is because, quite simply, I don't move.

Alice: Alas, poor Bob! Nature conspires against him.

Bob: Alas, poor Alice! She thinks that I be mad,
When all too well I know the madness lies
In her. So shall it be with George and Flo;
(When Di and Charles came by it was also so.)
(*Bob passes by Alice and recedes into the distance*)

Eve: O wicked Nature, to conspire 'gainst Robert
That all his gross and lamentable follies
Most undetectable thy tricks have rendered.

Alice: Sadder still, but for delusions twain
Bob has a most incisive, cogent brain.
Alas poor Bob! And Charles! And Di! and George!

Bob: (it From afar) Alas poor Alice! And Charles! And Flo! And George!

Flo: (*coming into view*) Alas poor Alice! and Bob! and Charles! and Di!
Chorus of relativists:
Such sorry discord need not be
If Absolutists had more sense:
So right in all their measurements,
So mad in their philosophy.