Physics 209: Assignment #7 (Final Paper on Relativity)

This is the final take-home exam for the relativity part of the course. As announced at the beginning of the course, will be graded on the conventional A,B,C,... scale and will count about the same as all the earlier assignments together, in determining your grade for the relativity part of the course. Although you have three weeks to work on it, one of the weeks is spring break (3/17-3/25), so think early on about how to budget your time. The assignment ought to require an amount of effort comparable to two ordinary assignments. I urge you to work on at least the first two parts before the break. (The third part will not make sense until you read Lecture Notes #11 on mass, energy, and momentum.)

Because this assignment serves as a take-home examination, you should not collaborate with others in working your way through the questions, as you may have been doing on the earlier assignments. If you are having difficulties understanding what you are asked to do, please feel free to consult with David Mermin, Trevor Olson, or Richard Helms by email, telephone, at office hours, or by appointment. General issues raised by these questions will also be discussed in the section meetings on March 12th and 26th. As always, the essays in which you try to convey your understanding of the questions should be entirely your own.

The Assignment is divided into three parts. At the beginning of each part I cite the relevant lecture notes, which you are welcome to reread. The calculations you are asked to do are not complicated. The challenge in each part is (a) to understand what it is I am asking you to do, (b) to do it, which should be easy once you get past (a), and (c) to write an essay that lucidly and graceful demonstrates that you have indeed understood the questions and how to answer them. Step (c) ought to be more difficult than steps (b) and (a).

Please read each Part (including footnotes) all the way through at least once (and perhaps twice) before concluding that you can't make sense of what I'm asking you to do. I have tried hard to spell things out.

NOTE: We will be starting on the second (chaos) part of the course shortly after spring break, using the Hewlett-Packard 20S programmable pocket calculators (about $35 at the campus store.) They tell me they have put in a reasonable supply of them, but don't wait until the last minute to get one. You will need it shortly after spring break. Try to get it before the break, so if they run out they will have that week to order more.

I.[1]

Figure 1 on page 4 shows two pictures. Each picture is drawn at a single moment of time in the frame of reference of a space station *(s)* (the black circle). The picture shows portions of two trains of numbered rockets — a grey *(g)* train and a white *(w)* train — moving uniformly past the station to the left and right. The numbers directly above each grey rocket and directly below each white one (like *:125*, *:175*, etc.) are the reading *in nanoseconds* (ns) of a clock in the center of that rocket. The clocks within each train of rockets are synchronized in the frame of the train.[2]

Let u be the speed of either train in the station frame, so the velocity v_{ws} of the white train in the station frame is u and the velocity v_{gs} of the grey train in the station frame is $-u$. Let v be the speed of one train in the frame of the other, so that the velocity of the white train in the frame of the grey train is $v_{\mathrm{wg}} = v$ and the velocity of the grey train in the frame of the white train is $v_{\mathrm{gw}} = -v$. Define the *foot* (f) to be 29.9792458 centimeters so the speed of light is exactly 1 f/ns. Take as another useful unit of length a *rocket* (r), which is the length of a rocket in its proper frame.

1. What is the speed u (in r/ns — rockets per nanosecond) of either train in the *station* frame? You can read this directly from the figure by noting that it is the same as the speed (in r/ns) of the station in the frame of either train. Draw a pair of pictures — two relevant fragments from the pair of pictures in Figure 1 — from which the speed of the station in the frame of either train can be extracted. Write a caption for the pair of pictures that makes it clear why the pictures show that the speed is what you say it is.

2. From your answer to Question 1, and the fact that in the station frame the clocks on either rocket are obviously out of synchronization by 1 ns/rocket, write a paragraph explaining how the $T = Du/c^2$ rule for clock asynchronization enables you to deduce the value of the speed of light c in r/ns. Since $c = 1$ f/ns by definition of the (Physics 209) foot, how many feet long is each rocket in its proper frame?[3]

3. Use the relativistic velocity addition law,

$$v_{\mathrm{wg}} = \frac{v_{\mathrm{ws}} + v_{\mathrm{sg}}}{1 + v_{\mathrm{ws}} v_{\mathrm{sg}}/c^2}, \tag{1}$$

[1] Reference: Lecture Notes 9, "Trains of Rockets."

[2] **Warning:** This is the real thing, not a simulation. The speed that plays the role of the speed of light is the speed of light itself, 1 foot per nanosecond. **Second Warning:** The speeds of the trains (as fractions of the speed of light) are not the same as they were in the computer program or in Assignment #5, nor does the speed of light (in rockets per ns) have the same value as it had in the computer program or Assignment #5 (in rockets per tick).

[3] Once you have the value of c in r/ns, the proper length of the rocket in feet follows immediately. It is a round number, appropriate for a one-man (or, perhaps more comfortably, a one-monkey) rocket.

to calculate from your answers to Questions 1 and 2 the velocity $v_{\rm wg}$ of the white rockets in the frame of the grey ones. In explaining your calculation be sure to make clear how the quantities appearing in the addition law (1) correspond to the numbers you found in Questions 1 and 2 (and state explicitly whether your answer is given in feet per nanosecond, or rockets per nanosecond.)

4. Extract two little pictures, one from each of the parts of Figure 1, that directly reveal the speed of the white rockets in the frame of the grey ones, and write a caption for your figure explaining how that speed can be read directly from the data in the two little pictures. If the speed your figure reveals is not the same as the speed you found in Question 3, something is wrong. Find your mistake(s) before you start writing.

5. Explain which of the numbers you have found in **1**-**4** enable you to calculate the slowing-down factor ($s = \sqrt{1 - v^2/c^2}$) for the clocks on the white train, as described in the frame of reference of the grey train.[4] Then extract two little pictures, one from each of the big pictures on page 4, that make it directly evident that the white clocks do indeed run slowly by this amount, according to people on the grey train, and write a caption for your figure that explains how this follows directly from your the data in your figure.

6. Suppose a photon moving to the right is at the space station in the upper half of Figure 1, and therefore opposite grey rocket 110 when its clock reads 127 ns.

(a) Suppose that the grey train has more rockets, numbered $113, 114, 115, \ldots$ in the upper picture and $114, 115, 116, \ldots$ in the lower picture, extending off to the right, each with its own clock, synchronized with the other grey clocks in the frame of the grey train. Using the value of the speed of light in r/ns that you found in **2** above, figure out another grey rocket that the photon will be at in the lower half of the Figure and what the clock at that grey rocket will read. Draw a picture showing the two grey rockets and the photon, with a caption indicating why .

(b) Add to the picture you drew in (a) the neighboring white rocket and its clock (assuming the white train extends to the right with rockets numbered $107, 106, 105, \ldots$ in the upper picture and $108, 107, 106, \ldots$ in the lower picture and write a caption explaining how the expanded picture illustrates the constancy of the velocity of light.

(c) Explain how to figure out directly from your picture in (b) together with the rest of Figure 1, how much faster than the white train the photon is going according to the station frame. Comment on why the number you come up with is consistent with the value the speed of light must have in the station frame.

[4] If what is inside the square root is not a perfect square, then you have made a mistake. Go back and correct it.

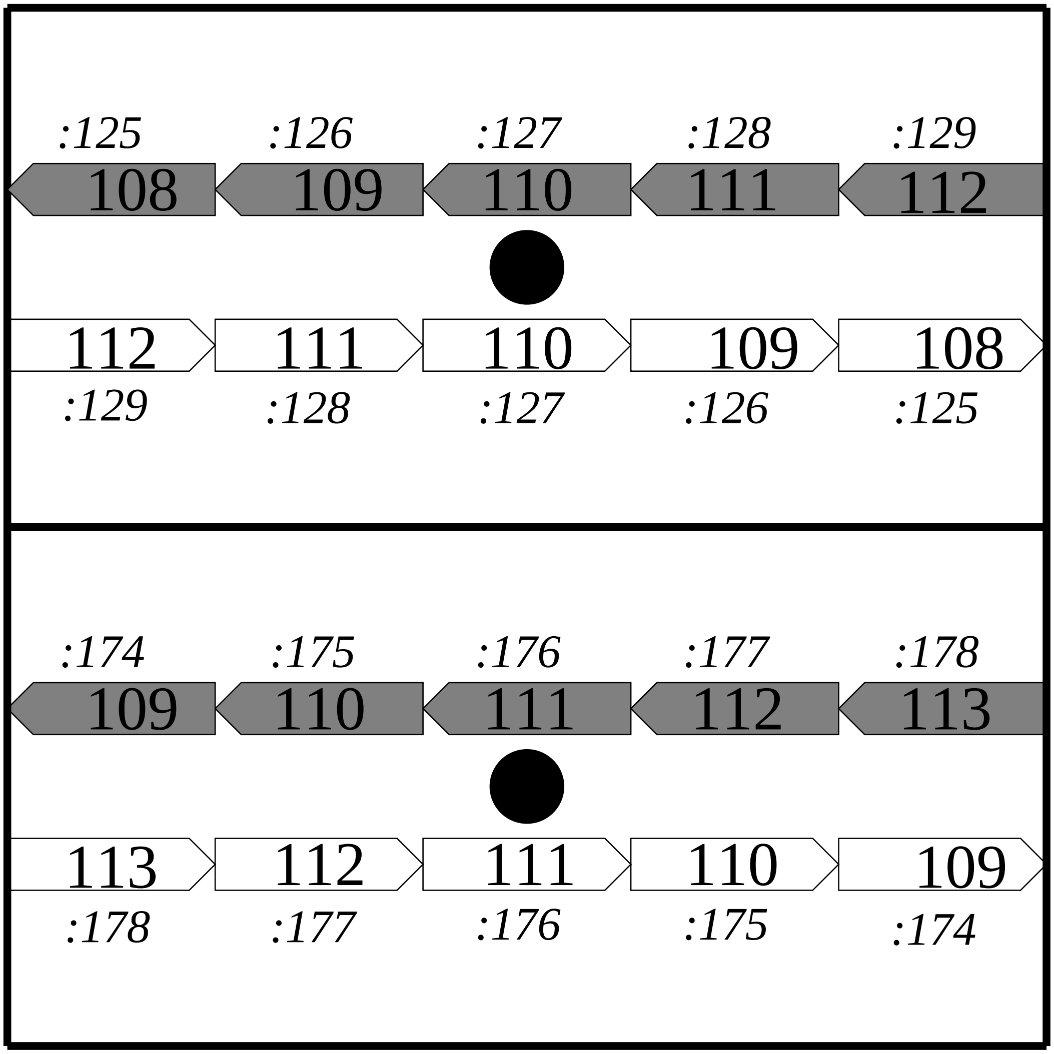
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Figure 1.

II.[5]

The interval between two time-like separated events is the time between them in a frame of reference in which they happen in the same place. The interval between two space-like separated events is the distance between them in a frame in which they happen at the same time. In a space-time diagram the squared interval between two events can be taken, in suitable units,[6] to be the area of the rectangle of photon lines that has the events at opposite vertices.

It turns out that the interval between two events can be measured in a simple way by a person who is present at just one of the events. All that person needs is a single clock and appropriate light signals to and from the other event. This part of the Assignment discusses how this is done. (You will be asked to provide captions for the space-time diagrams that illustrate this procedure.)

Let Alice (whose proper frame is inertial) and her clock be present at the event E_1. Call the reading of Alice's clock when E_1 occurs 0.[7] Alice watches for the second event E_2 through a telescope. She notes the time T on her clock at the moment she *sees* E_2 take place. She also notes the time t that Bob, who is present at the event E_2 and watching Alice's clock through his own telescope, *sees* Alice's clock reading at the moment the second event takes place.[8] The squared interval between the events turns out to be given simply by

$$I^2 = |tT|. \tag{3}$$

This not immediately obvious fact is illustrated (and proved) by the space-time diagrams in Figures 2 and 3 on pages 6 and 7. The figures are complete — i.e. it is not necessary to add any lines to them to make the point. One figure applies to a pair of space-like separated events and the other, to a different pair of time-like separated events.

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[5] Reference: Lecture Notes 10, Space-Time Diagrams. Lecture Notes 8, Invariance of the Interval between Events,y contain the basic facts about space-time intervals, but everything you need to know about them is stated concisely below.

[6] The scale factor that connects squared interval to area in the diagram is half the frame-independent product of scale factors, $\frac{1}{2}\lambda\mu$, but you do not need this additional piece of information to answer the questions that follow.

[7] This is a convenient choice, because one can then interpret the clock readings at other times as the *difference* between those readings and the reading at the event E_1.

[8] Warning: Do not assume that t is necessarily a positive number. Alice can know what t is because Bob can write t down as he reads Alice's clock through his own telescope, and Alice's telescope can be powerful enough for her to read what he writes. Alternatively, Bob can be replaced by a giant mirror at the second event which reflects the image of Alice's clock reading t back to Alice.

Write captions for each figure explaining how it illustrates this story.[9] The only quantitative properties of the interval you need to establish the relation (3) are those that were summarized in the first paragraph of Part II, above.[10] To explain how (3) follows from the figures you must think not only about the interval between E_1 and E_2, but also the intervals between E_1 and the two events consisting of Alice's clock reading t and reading T.

Your caption should indicate which figure applies to which case, and should identify which part of the figure is the event E_1, which part is the event E_2, which line is the world-line of Alice and her clock, which dashed line is the trajectory of the photon that enables Alice to see the clock at E_2, which dashed line is the trajectory of the photon that enables Alice to the reflection of her own clock in the mirror, which rectangle of photon lines has an area equal to the squared interval between the events, which rectangles of photon lines have areas proportional to the squares of the times t and T, and why the areas of the rectangles are related in such a way as to establish Eq. (3).[11]

No elaborate analysis is needed. The relation (3) is a simple consequence of the geometric relations revealed in the figures, when those figures are properly interpreted. The subtle part is providing the interpretation. *If you cannot figure out how to extract relation (3) from the figures, you can still get substantial partial credit by simply writing a caption that states as clearly as you can how the diagrams illustrate the story.*[12]

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[9] If it is worded carefully, a single caption can work for both figures, provided it indicates which figure applies to the space-like separated pair of events and which to the time-like separated pair.

[10] It is also necessary to take explicitly into account the (obvious) fact that the time between two events in a frame in which they happen at the same place is just the change in reading between the two events of a clock that is present at both of them.

[11] Feel free to use terms like "thin solid line", "thick solid line", "thin dashed line", "thick dashed line", "white circle", "grey circle", "right", "left", "upper", "lower", etc. You can identify rectangles by specifying the events that take place at diagonally opposite vertices.

[12] It might help to note that the distances labeled a, b, fa, and fb and the light dashed lines are only relevant to establishing (3).

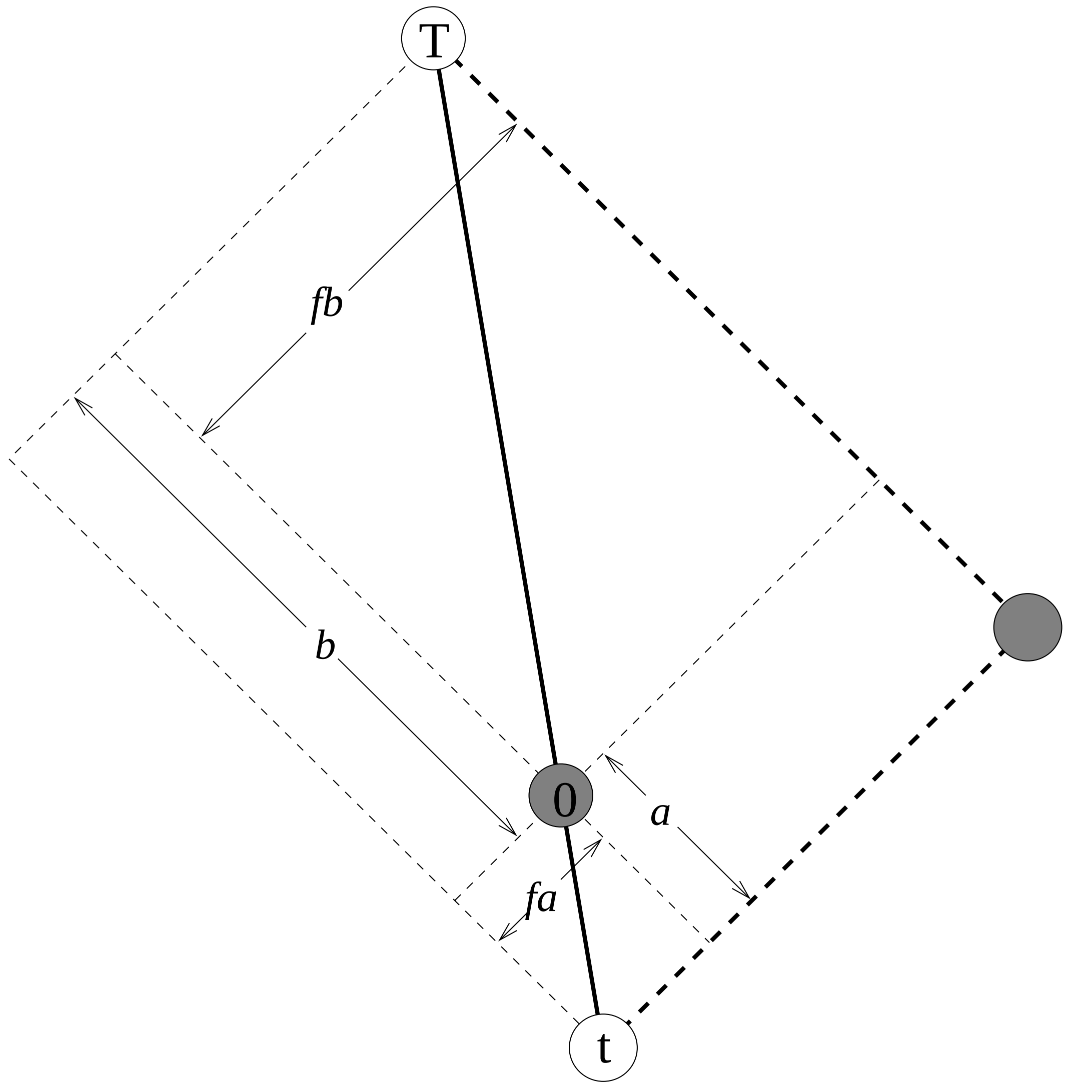

Figure 2.

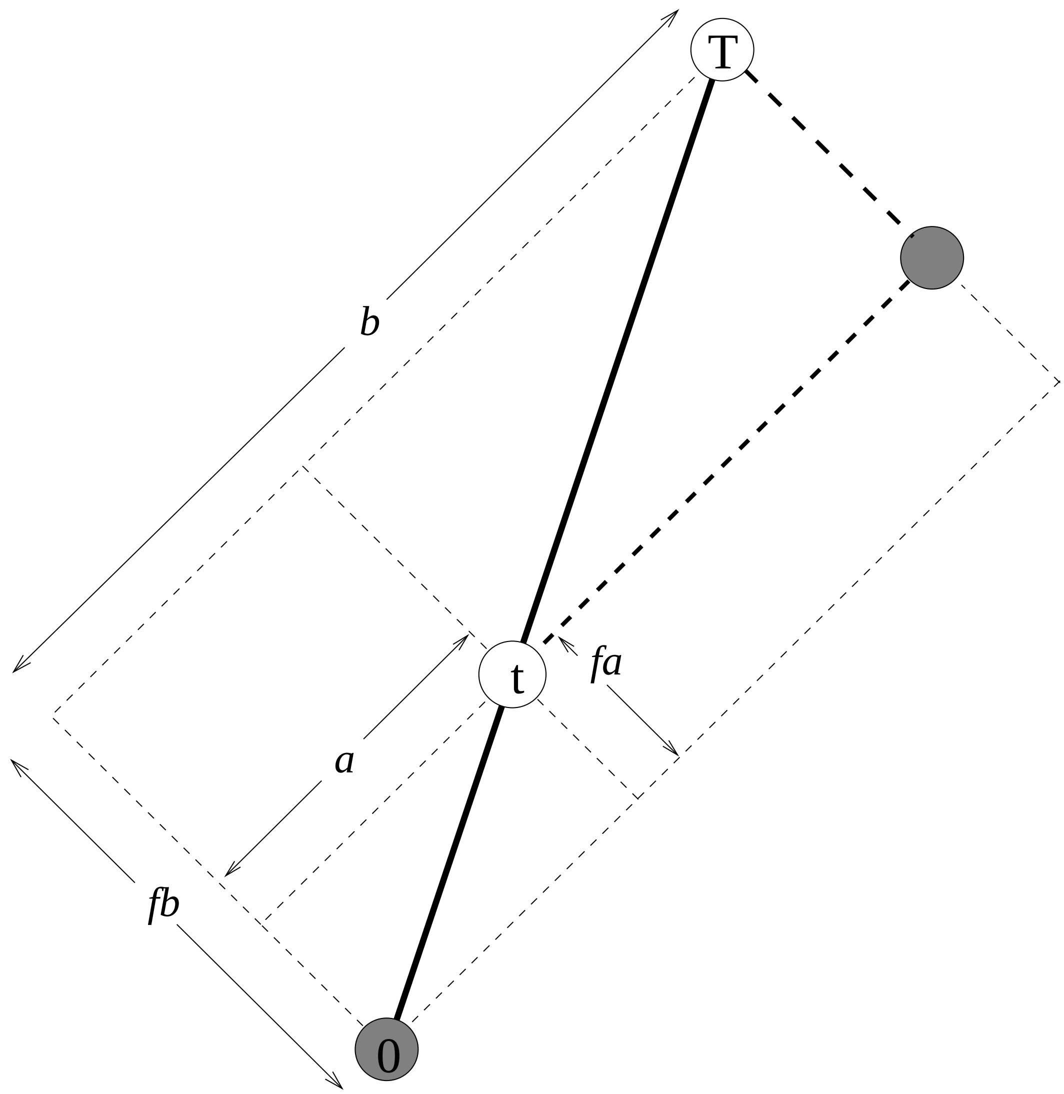

Figure 3.

III.[13]

A stationary particle of mass M undergoes a radioactive decay in which it emits a photon, and turns into a particle of mass m. This is illustrated in Figure 4 on page 10.

(1) Find an expression for the energy ω of the photon in terms of the masses M and m of the particle before and after the decay (and, of course, the speed of light c).

(2) Find an expression for the speed u of the particle after the decay in terms of the masses M and m of the particle before and after the decay (and the speed of light).

(3) Comment on why your answers to (1) and (2) make sense when $m = M$. Comment on why they are reasonable when $m = 0$. (If they do not make sense or are not reasonable perhaps you have made a mistake.)

The way in which these questions are to be approached is quite similar to the way in which the absorption of a photon by a ball is discussed in Appendix B of Lecture Notes #10 on mass, momentum, and energy.[14] All you have to know are the following facts:

1. The mass m, energy, e, and momentum p of any particle are related by:

$$e^2 = (pc)^2 + (mc^2)^2. \tag{2}$$

2. The velocity u, energy e, and momentum p of any particle are related by:

$$u = pc^2/e. \tag{3}$$

3. A photon is a particle whose mass is 0.

4. Total energy is conserved in a decay:

$$E_b = E_a \tag{4}$$

where E_b is the total energy of all the particles present before the decay and E_a is the total energy of all the particles present after the decay.

5. Total momentum is conserved in a decay:

$$P_b = P_a \tag{5}$$

where P_b is the total momentum of all the particles present before the decay and P_a is the total momentum of all the particles present after the decay.

[13] Reference: Lecture Notes 11, "Mass, Momentum, and Energy."

[14] Your discussion should be at a literary level comparable (or superior) to that of Appendix B.

Here a suggestion for how to proceed:

Note (a) the very simple relation between the momentum k and energy ω of the photon, (b) the very simple relation momentum conservation gives between the momentum k of the photon and the momentum p of the particle after the decay, (c) the relation energy conservation gives between the energy e of the particle after the decay, the energy ω of the photon, and the mass M of the particle before the decay, and (d) the simple relation between the energy e, momentum p, and mass m of the particle after the decay. You should indicate which of the facts 1-5 you use in the course of establishing (a)-(d). Using (a), (b), and (c) you can rewrite (d) as an equation in which only the photon energy, the two masses, and the speed of light appear, which can easily be solved to answer question (1).

To answer question (2) use the simple relation between the energy e of a particle, its momentum p, and its speed u. Relations (a)-(c) above and your answer to question (1) are then enough to give you u in terms of the two masses.

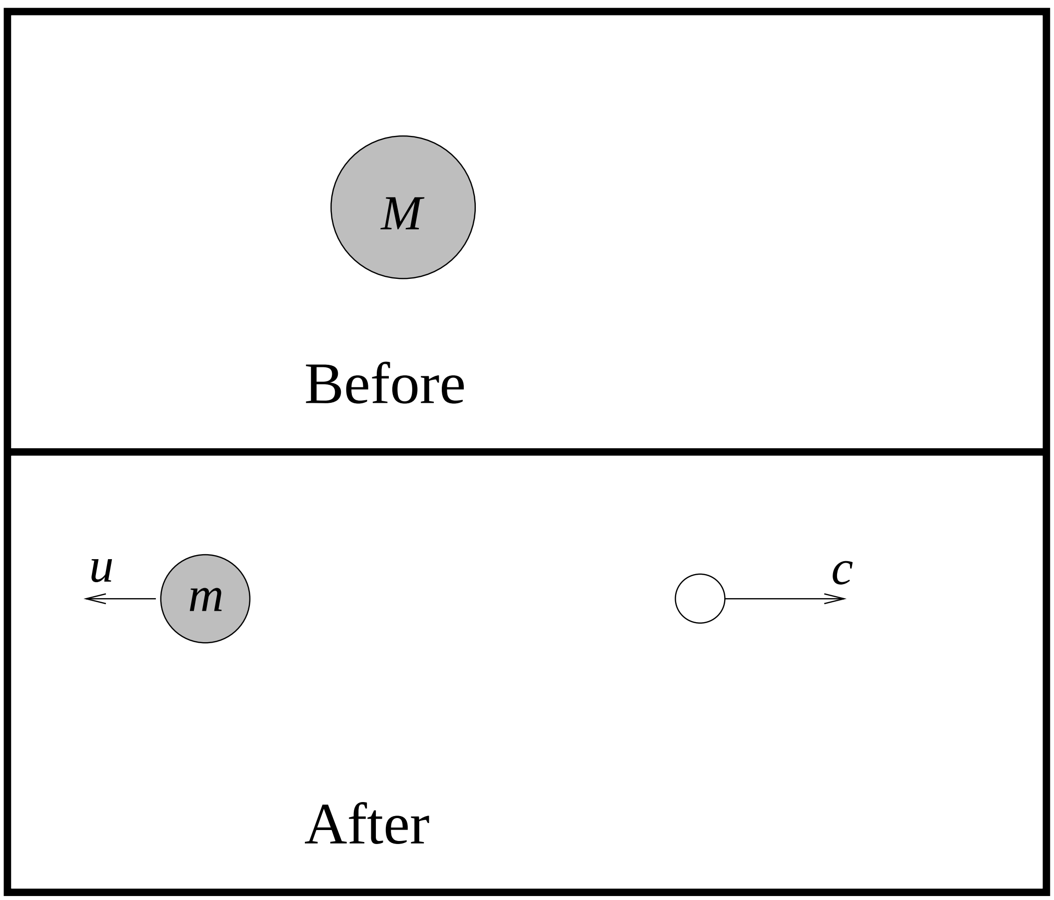

Figure 4.