

## Physics 209: Assignment #11

We continue to explore features of the sine map,  $x \rightarrow \frac{a}{4} \sin \pi x$ . Your reply to the questions that follow should be in the form of an essay that not only reports the relevant numerical results but gives some indication of how you went about getting them and the adventures or misadventures you had along the way. For your convenience, the program that iterates the sine map is reproduced here:

| Key name         | Key code | Comments  |
|------------------|----------|---|
| [y] <b>LBL E</b> | 61 41 E  | <i>Program E. Starts with x in display,</i>         |
| (                | 33       |   |
| (                | 33       |   |
| ×                | 55       | <i>multiplies it by</i>                             |
| [y] π            | 61 22    | <i>pi,</i>  |
| )                | 34       |   |
| <b>sin</b>       | 23       | <i>and evaluates sin(πx)</i>                        |
| ×                | 55       | <i>which is then multiplied by</i>                  |
| <b>RCL 9</b>     | 22 9     | <i>a</i>  |
| ÷                | 45       | <i>and divided by</i>                               |
| <b>4</b>         | 4        | <i>4</i>  |
| )                | 34       | <i>to get <math>\frac{a}{4} \sin(\pi x)</math>.</i> |
| [y] <b>R/S</b>   | 26       | <i>Displays result.</i>                             |
| [b] <b>GTO E</b> | 51 41 E  | <i>Iterates.</i>                                    |

You were asked in the last Assignment to find the value  $a'_0$  of the control parameter  $a$  at which the non-zero fixed point splits into a 2-cycle. To the full accuracy of the calculator I find that value to be  $a'_0 = 2.87984673191$ . In case you are curious, the value  $a_0$  at which the non-zero fixed point first appears (and the fixed point at 0 becomes unstable) is  $a_0 = \frac{4}{\pi} = 1.27323954474$ . Notice how different these are from the corresponding values,  $a_0 = 1$  and  $a'_0 = 3$  for the logistic map.<sup>1</sup>

### I.

1. Find the value  $x^*$  of the fixed point when  $a = a'_0$  by iterating from any initial value of  $x$ , speeding up the process by averaging the results of successive iterations. First simply

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<sup>1</sup> On the other hand for either map 0.5 is easily seen to be a fixed point when  $a = 2$ , so both maps have a superstable fixed point at the same value of  $a$ .

iterate for a while until you get to a stage where the values of  $x$  are going up and down on successive iterations. Comment on how rapidly the upper and lower values seem to be approaching each other. Then speed up the process by averaging the results of successive iterations. You get such an average by doing  $+ \mathbf{R/S} = \div \mathbf{2} =$ . Pause, in the course of doing this, to see whether you are hanging up on bogus 2-cycles. How big a bogus 2-cycle can you find? By a bogus 2-cycle I mean a pair of values, quite close together, that the calculator cycles between not because the map has converged to a 2-cycle, but because the change from one pair of iterations to the next pair is smaller than the accuracy of the calculator. By the size of the bogus 2-cycle I mean the amount by which the two points on the “cycle” differ. You might think that only the last (12th) digit would change in such bogus cycles, but if the rate of convergence to the 2-cycle is extremely slow, as it is at the bifurcation value of  $a$ , quite a few of the last digits can change. The test that a 2-cycle is bogus is that if you move the two points a little closer together and then iterate, their separation will not grow (bringing you back to the original 2-cycle) but instead you will quickly hang up at a smaller but still bogus 2-cycle.

It’s reasonable to assume that the fixed point is extremely close to the average of the two values at which the calculation gets stuck. If this is true then once you reach a bogus 2-cycle the next round of averaging should take you right in to the fixed point. Check to see if this does give you a fixed point  $x^*$ . Comment on how reproducible is the value for  $x^*$  that you get by this method, by doing it over again several times, starting with widely different initial values of  $x$ .

2. When  $a$  is a little above  $a'_0$  (say 2.8799) find the two points on the genuine 2-cycle to five- or six-place accuracy. I would recommend starting some distance away from the fixed point you found in 1, and then helping things along when you start getting close to the 2-cycle by adjusting the values in the display up or down by simply reentering new values. Provide some evidence that the values you quote are a genuine 2-cycle, by noting what happens if you start with a value of  $x$  that is slightly bigger than the larger of the two values and what happens if it is slightly smaller. As in class, the most effective way to do this is to focus on only one branch of the 2-cycle, by looking only at the doubly iterated (push the **R/S** button twice) map. Beware of bogus 2-cycles, whose two values are suspiciously close together! You can get stuck on them if you start with a value too close to the unstable fixed point, since near the unstable fixed point values move away from each other *extremely slowly*.

## II.

3. Using techniques analogous to those we used in class, examine values of  $a$  larger than  $a'_0$  to find the value  $a''_0$  at which the 2-cycle gives way to a 4-cycle. Get  $a''_0$  to at least 4 place accuracy. Here and in part 4 below, be sure to give some concise but convincing numerical evidence that your answer is correct. For example if you wanted to demonstrate that  $a''_0 = 3.4567$  (a number I just made up for the sake of the illustration) you would have to give evidence that 3.45675 definitely gives a 4-cycle and 3.45665 definitely gives a

2-cycle).

4. Find to at least 4 place accuracy the value  $a_0'''$  at which the 4-cycle gives way to an 8-cycle. It turns out that  $a_0'''$  is considerably closer to  $a_0''$  than  $a_0''$  is to  $a_0'$ . Note, for future reference, the value of the ratio  $\frac{a_0' - a_0''}{a_0''' - a_0''}$ . (It should be a crude approximation to the famous Feigenbaum number: 4.669..., about which we will have more to say.)

### III.

5. Find to at least 6-place accuracy the value  $a_2$  of the control parameter  $a$  for which the 2-cycle is superstable. (I was able to get it to 9 place accuracy in about 15 minutes. Except for dealing with the irritation of the three hidden places, it would not have been hard to find it to the full 12-place accuracy of the calculator.) A superstable cycle for the sine (or logistic) map is one that has among its values the value  $x = 0.5$ . Here is a procedure that efficiently leads to the superstable value of  $a$ : Start with a value of  $a$  somewhere between  $a_0'$  and  $a_0''$ , let  $x = 0.5$ , iterate twice, and see if the new value of  $x$  is more than or less than 0.5. If it is more than [or less than] 0.5 then hunt around for another value of  $a$  in the same range for which 0.5 twice-iterates into a value less than [or more than] 0.5. You have now sandwiched  $a_2$  between those two values. Next try a value of  $a$  about half way between. Depending on whether 0.5 twice-iterates to more or less than 0.5 you will have now sandwiched  $a_2$  into either one or the other half of the old range. In this way you can keep cutting the range in which  $a_2$  lies about in half (and will be finding that with each successive halving of the uncertainty, 0.5 twice-iterates into numbers that are closer and closer to 0.5). I found that to do this efficiently it was useful to make two parallel columns of numbers on a piece of paper: numbers for which 0.5 twice-iterates into a little more than 0.5 and numbers for which it twice-iterated into a little less. As the columns get longer the numbers at their bottoms get closer and closer together. On the piece of paper that got  $a_2$  to nine decimal places, I ended up with only about ten entries in each column.

6. Find to at least 6-place accuracy the value  $a_4$  of the control parameter  $a$  for which the 4-cycle is superstable.