Thursday, April 12 2001 **Due April 19**

Physics 209: Assignment #10

We will be playing in class with the logistic map:

$$x \to ax(1-x), \tag{1}$$

with x between 0 and 1, and a between 0 and 4. There is a qualitatively similar but quantitatively quite different map that you can use to conduct similar investigations at home. We shall call it the sine map:

$$x \to \frac{a}{4}\sin(\pi x) \tag{2}$$

with x between 0 and 1 and a between 0 and 4.

In terms of the fly walking up the wall of a circular dome of radius 1 on a great circle passing through the top of the dome, $\sin(x)$ is defined as the height the fly has reached when it has gone a distance x along the circle.[1] Thus $\sin(0) = 0$ because the fly starts on the floor of the dome; $\sin(\frac{\pi}{2}) = 1$ because the fly is at the top after going a quarter of the full distance (2π) around the circle, and $\sin(\pi) = 0$ because when the fly has gone half way around the circle it is back on the floor on the other side. The fly and the dome are useful for defining $\sin(x)$, but play no role in interpreting the sine map, which you should view as simply an abstract modification of the logistic map, which is easy to program since the calculator has a built in sine-calculating routine. The point of this and subsequent assignment will be to note the similarities and differences of the behavior of the two maps under iteration.

The two maps are qualitatively similar in many ways. Both $ax(1-x)$ and $\frac{a}{4}\sin(\pi x)$ share the following properties:[2] (1) they are both 0 when $x = 0$; (2) they are both 0 when $x = 1$; (3) they both take on their largest value between 0 and 1 at $x = \frac{1}{2}$, where they have the value $a/4$; (4) they are both symmetric about $x = 1/2$ — i.e. their value at x depends only on how far x is from $\frac{1}{2}$ and not on whether it is greater than or less than $\frac{1}{2}$. In their more detailed behavior, however, they are quite different. For example the graph

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[1] See the figure on the last page. It is one of the curiosities of mathematics that one writes "sin" in formulae and "sine" in ordinary prose. This is because "sin" is an abbreviation for "sine", just as "cos" is an abbreviation for "cosine" and "tan" is an abbreviation for "tangent". The 3-letter naming rule seems to have triumphed over the fact that one rarely abbreviates a word by chopping off a single letter.

[2] Check each of these for yourself.

of $y = ax(1-x)$ climbs more steeply away from $y = 0$ near $x = 0$, then the graph of $y = \frac{a}{4}\sin(\pi x)$. So while one might find qualitative similarities in the behavior of the two maps, one would be surprised to find precise quantitative similarities.

Since sine is hard–wired into the calculator and controlled by a single **SIN** key (23) the program to iterate the map is simple to write and runs fast.[3] The program assumes that the control parameter a has been stored in location 9, and the initial value of x (a number between 0 and 1) has been entered in the display. If you have no programs you are interested in keeping then clear the program memory before entering the program with [y] **CLPRGM** (61 31). If you do have some programs you wish to keep — you will certainly want to keep the logistic-map program — then you can enter this one after the last instruction currently in program memory, or before any of them. To enter it at the *beginning* of program memory, do [b] **GTO . .** (51 41 73 73)), and then enter the new program. To enter it *after* everything else, do **GTO ..** and then do blue up-arrow (51 43) to get to the last location you are currently using. You can then enter the program at that point. It makes no difference which of these options you exercise. I have labeled the sine-map program with **E** under the assumption that you have no other program labeled **E**.

Here is the program for the sine map:

	Key name	**Key code**	**Comments**
[y]	**LBL E**	*61 41 E*	*Program E. Starts with x in display,*
	(	*33*	
	(	*33*	
	×	*55*	*multiplies it by*
[y]	π	*61 22*	*pi,*
	)	*34*	
	sin	*23*	*and evaluates* $\sin(\pi x)$
	×	*55*	*which is then multiplied by*
	RCL 9	*22 9*	a
	÷	*45*	*and divided by*
	4	*4*	*4*
	)	*34*	*to get* $\frac{a}{4}\sin(\pi x)$.
[y]	**R/S**	*26*	*Displays result.*
[b]	**GTO E**	*51 41 E*	*Iterates.*

Some reminders: You go into program mode [b] **PRGM** [51 26]) only to enter programs. Once you have entered the program you leave **PRGM** mode by doing [b] **PRGM** again,

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[3] If you have forgotten, see Assignment #8 for instructions on how to enter a program.

which gets you back into normal mode. Everything that follows assumes you are in normal mode—i.e. that you have already entered the program. The program you have entered assumes that you are operating the calculator in "radian mode". Never mind what that is.[4] You get into radian mode by pressing [y] **RAD** (61 24). The word RAD will then appear in the display where it will remain until you do something to get out of radian mode.[5]

To run the program you must store the parameter a in location 9. Enter a (a number bigger than 0 and less than 4) and do **STO 9** ("store the displayed number in location 9"). Clear the display (**C**) and do **RCL 9** ("recall from location 9 and put in display") to make sure you've entered your a correctly. Then enter in the display a number x (also bigger than 0 and less than 1) that you want the program to act on and press **XEQ E**. The program will then put in the display $\frac{a}{4}\sin(\pi x)$. You can then iterate as many times as you want simply by pressing **R/S** for each iteration. You will find that for small enough values of a (for example $a = 0.5$) no matter what x you start with (always between 0 and π) successive iterations approach the fixed point 0.

In these problem sets we will do for the sine map the same kinds of experiments we are doing in class for the logistic map.[6]

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[4] It has to do with the units in which angles are measured. Degrees are the units of the Babylonians; radians are the units of God. The radian measure of the angle at which you, at the center of the dome, have to look up to see the fly is the distance up the roof of the dome the fly has walked.

[5] There is no reason ever to do this unless you want to measure angles in degrees, in which case you must press [y] **DEG** (61 23).

[6] In the investigations that follow you should experiment between running the calculator in in the **ALL** mode (do [y] **ALL** (61 34)) or having it display just 9 decimal places by doing [b] **FIX** 9. (51 33 9). The **ALL** mode has the possibly irritating feature of writing a number like .001234579 in the ugly form 1.234579E-3 (where the E-3 at the end means put a lot of zeros before the number and then shift the decimal point three places to the left.) In the 9-place mode the calculator will show a number like 0.00045876 in that form, rather than as 4.5876E-4. But it has the disadvantage of showing a number like 0.000009999925… as 0.000010000, and only revealing that the number has been rounded *up* rather than *down* to 0.000001… when you do [b] **SHOW** (51 73). The **ALL** mode rewards you for putting up with the ugly E-6 by showing you directly the important digits, 9.999925. This becomes useful when you want to find out whether running the program on a very small number gives you a number that is very slightly smaller or very slightly larger. But even in **ALL** mode occasions may arise when you may want to see all 12 digits (which you often can't because digits are sacrificed to make room for things like E-2). You can do this by doing [b] **SHOW**. Except when dealing with very tiny numbers, my own preference is to use the [b] **FIX** 9 mode most of the time, using the [b] **SHOW** button when I want to see the last three digits.

Part I

As with the logistic map, there is a critical value a_0 of the control parameter a, below which the sine map has 0 as a stable fixed point. Above a_0 the fixed point 0 is unstable. There is a range of a above a_0 for which the iterations converge to a non-zero fixed point. The value of a_0 for the logistic map is 1. For the sine map it is roughly 25% larger.[7] Here is how to pin down a_0 numerically:

Try a value of a a little bit bigger than 1 — say $a = 1.1$. Store it in location 9 (enter 1.1 in the display and do **STO 9**)[8] Take a starting value of x between 0 and 1, run the program over and over,[9] and convince yourself that x is converging to a stable fixed point at zero.[10]

Now repeat the procedure for a larger value of a, say 1.5. Describe the numerical evidence that x is converging to a non-zero fixed point. What is the value of the fixed point x^* when $a = 1.5$? (It is not hard to find it to the full 12 place accuracy of the calculator.)

You have now established that a_0 is somewhere between 1.1 and 1.5. Next try $a = 1.3$. If numerical experimentation persuades you that 0 is a stable fixed point then you know that a_0 must be bigger than 1.3 and therefore between 1.3 and 1.5. But if numerical experimentation persuades you that 0 is an unstable fixed point then a_0 must lie between 1.1 and 1.3. Now you can refine things further, trying either $a = 1.2$ or $a = 1.4$ depending on what you have just learned.

In this way you can home in on the critical value a_0. I found without much trouble that I could find a_0 to five digits with confidence that the final digit was correctly rounded up or down. Find a_0 to at least this accuracy. (Feel free to pin it down even more precisely.) Write a brief description of what you had to do to get your value of a_0 mentioning, for example, how many upper and lower bounds you had to investigate. Give some direct numerical evidence that the value you quote for a_0 is right, by demonstrating that if you lower the final digit you quote by one, the fixed point at zero is stable, while if you raise it by one the fixed point at zero is unstable.

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[7] If you think a little about the figure on the last page and are used to that kind of reasoning, it is not difficult to figure out the exact value of a_0, but I'm not asking you to do this. Indeed, even if you do figure out its exact value, the point of this exercise is for you to demonstrate how well you can determine that value by the kind of numerical investigation I'm asking you to undertake.

[8] I will stop reminding you of this. Every time you want to try out a new value of a you begin by storing it in location 9.

[9] Do **XEQ E** the first time; after that, unless you run another program inbetween or get hung up on an error, its enough to do **R/S**. I will stop reminding you of this too.

[10] Zero is always a fixed point of the sine map. If it is stable then when you start at a small but non-zero value, you get closer to zero. If it is unstable, when you start at a small non-zero value and iterate you get farther away.

The test for a given value of a is rather simple. Once you've found that iterating from a significant non-zero value gets you down below 10^{-12} (0.000 000 000 001, or 1.E-12) you can be quite sure your value of a is one for which zero is a stable fixed point.[11] And once you've found that iterating from a very small non-zero value of x leads to a *larger* value, you can be quite sure your value of a is one for which 0 is not a stable fixed point.

Part II.

As a increases above the value of a_0 you found in Part I, the non-zero fixed point eventually becomes unstable, splitting up into a stable 2-cycle. Here is a way to find the value of a (call it a_0') at which this bifurcation of the fixed point to a 2-cycle takes place. Go up from your value of a_0 (rounded off to just a single digit past the decimal point) in steps of 0.2 until you reach a value of a for which iteration clearly converges to a 2-cycle. Write a paragraph comparing the different kinds of approach to the fixed point you observed for the different values of a before you got to the 2-cycle, commenting on whether convergence to the fixed point is fast or slow, and whether the values of x after the first few iterations grow, shrink, or jump back and forth (oscillate) under subsequent iterations.

The first value of a you find for which convergence is to a 2-cycle is an upper bound for a_0'. Report the value of this first upper bound, and give the evidence that the 2-cycle is stable. Note that it is not enough to show that the calculator is hopping back and forth between two numbers, because if the numbers are getting closer and closer together, even rather slowly, you could be observing a very slow process of convergence to a fixed point. To convince yourself that this is not happening you must find a case in which the calculator hops between two numbers that get farther and farther apart under further iterations, and not closer and closer together.

Now close in on a_0' as you did on a_0 by seeing whether you get a fixed point or a 2-cycle at a value of a half way between the last one that gave you a fixed point and the first one that gave you the 2-cycle. This will cut the possible range for a_0 in half. Next, you can again try a value of a in the middle of that new smaller range, and in this way home in on a_0'. You should again get a_o' to at least five place accuracy.

As you get closer and closer to a_0' it will become harder and harder to tell the fixed point from the 2-cycle. When a is just a tiny amount above a_0', the two numbers of the 2-cycle will be rather close together. And when a is a tiny bit less than a_0' convergence to the fixed point will be oscillatory and so slow that it may be hard to distinguish from convergence to a 2-cycle.

A definitive test that you have a 2-cycle is to run the program for a point somewhere between the two points of the alleged 2-cycle. If on alternative runs you find your answers are then getting farther and farther apart, trying to grow back to the alleged 2-cycle you

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[11] You would have to be only an extremely tiny amount above a_0 if what was actually happening was convergence to a non-zero fixed point smaller than 10^{-12}.

started from, then you did indeed have a legitimate 2–cycle. If, on the other hand, the number in the display now jumps back and forth between two even closer values, then you merely had extremely slow convergence to a fixed point.[12]

If you think you are oscillating in to a fixed point but with maddening slowness, you can speed up convergence by jumping to a new value of x midway between the two you are hopping between. To get a point that is half way between the value currently in the display and the next one, simply do: **+ R/S = ÷ 2 =**. You may have to do this several times. There are two things to be wary of when you do this. (1) Beware of encountering bogus 2-cycles in the course of this process; (2) Beware of forcing a genuine 2-cycle to look like a fixed point by excessive use of this averaging procedure. It takes some playing around to acquire the appropriate instincts. You are encouraged to report your adventures and misadventures.

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[12] **Warning:** if you really manage to get quite close to a'_0 you may find that convergence to the fixed point has become so slow that the calculator gets stuck at a bogus 2-cycle, because the tiny amount by which the two values get closer together with two more iterations is less than the 12-place accuracy of the calculator. Beware of bogus 2-cycles! They can show up even when three or four of the final digits are still hopping back and forth. The test of a genuine 2-cycle is always that if you start at a point closer together than the points of the alleged 2-cycle, successive iterations takes you back out to those points. If the 2-cycle is bogus and you try this you will get hung up at a smaller (but still bogus) 2-cycle.

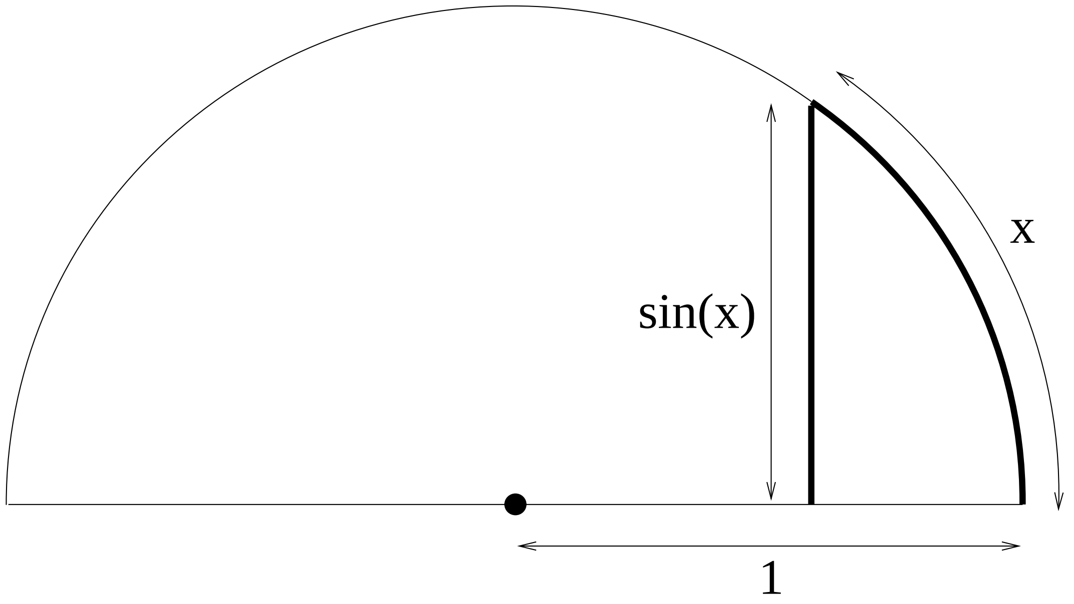

Figure: Definition of sin(x).

Express all lengths as multiples of the radius of the semicircle (so that the radius itself has length 1). If x is the length of the heavy part of the circle, then $\sin(x)$ is the length of the heavy vertical line. So $\sin(x)$ is the distance of the fly above the floor when it has gone a distance x up the circular roof. Note that it follows from this that $\sin(0)$ is 0. And since the length of the entire semicircle is π, $\sin(\pi)$ is also zero. But $\sin(\frac{1}{2}\pi)$ is 1.

Note that if x were very small, the heavy part of the circle would be barely distinguishable from the heavy vertical line. This tells us that when x is very small, $\sin(x)$ is very close to x. Check this out with your calculator.