9. Trains of Rockets

In this Chapter we shall introduce no new technical concepts, but shall examine a particularly simple way to explore how a disagreement about whose clocks are synchronized leads to all the relativistic effects we have been examining: the slowing down of moving clocks, the shrinking of moving sticks, the relativistic velocity addition law, the existence of an invariant velocity, and the invariance of the interval.

We do this by examining two frames of reference from the point of view of a *third* frame in which the first two move with exactly the same speed, but in opposite directions. We take the third frame to be the proper frame of a space station, represented by the black circle in Figure 1.[1] The other two frames are the proper frames of two trains of rockets: a grey train, moving to the left in the four parts of Figure 1, and a white train, moving to the right with the same speed as the grey train.

Figure 1 shows the station and the two trains of consecutively numbered rockets at four different moments of time, as described from the point of view of the station frame. The station is in the same place in all four parts of the figure, while each train has moved an additional rocket's length from one part of the figure to the next. The three numbers preceded by a colon (e.g. :006) adjacent to each rocket represent the reading of a clock carried by that rocket. Think of each clock as being at the center of its rocket, right next to the number of the rocket.

Notice that the clocks on either train of rockets, in each of the four parts of the figure, are not synchronized: as you go towards the rear of either train the clocks get further and further ahead, the asynchronization being exactly two temporal units (which we shall call "ticks") per rocket of separation. This is in accord with the station-frame rule that if clocks have been synchronized in the train frame then they are out of synchronization in the station frame, a clock in front being behind a clock in the rear by $T = Du/c^2$ where, D is the distance between the clocks in their proper frame, and u is the speed of the train in the station frame. If we take as our unit of length the proper length of a rocket, then evidently the figure has been drawn for a value of u for which the asynchronization is

$$u/c^2 = 2 \text{ ticks per rocket.} \tag{9.1}$$

[1] I have put the figure on the last page, to make it easier to find in the many subsequent references to it.

You can take two attitudes toward Figure 1. You can imagine that both trains are moving with such prodigious speeds and the tick is such a tiny unit of time and the clocks so very precise, that the asynchronization depicted in the Figure is the genuine relativistic effect: u/c nanoseconds of asynchronization for every foot of separation.

Alternatively, and more entertainingly, one can take the view that the rockets are moving at perfectly feasible speeds — perhaps several feet per millisecond — and the clocks are quite ordinary clocks, ticking off seconds with good but not phenomenal precision, which have, however, been deliberately set out of synchronization by people in the space station. Under this reading of the figure, the station people want to test what kind of conclusions people on either train will arrive at using unsynchronized clocks, if they fail to realize that their clocks are not synchronized. So before the trains start to move, but after people have been locked into their rockets, the station people give the occupants of each rocket a clock. But they secretly set the clocks behind by two ticks per rocket as they move from the rear towards the front of the train, distributing the clocks. They also carefully arrange things so that people in different rockets cannot communicate with the people in other rockets of their train to compare notes on what their clocks read. The space station people lie to the occupants of each train, falsely assuring them that clocks in different rockets are synchronized.

Once the trains are set into motion, the only information people from either train can collect, is about what is going on in their immediate vicinity. In particular when two rockets are exactly opposite each other[2] then the occupants of either rocket can note the number and clock reading of the other rocket (as well, of course, as their own). Such information can be summarized in a little figure — a photograph that people on either of the rockets might have taken — that shows just those two adjacent rockets.

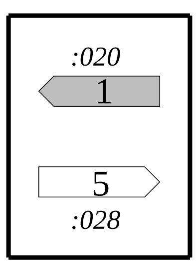


Figure 2

Figure 2, for example, is a fragment of part (d) of Figure 1, showing a picture that the

[2] For example in part (a) white and grey rockets 0 are directly opposite, in part (c) grey rocket 1 is directly opposite white rocket 3, and so on.

occupants of grey rocket 1 or white rocket 5 might have taken at the moment they were directly opposite, depicted in Figure 1 (d).

Contemplating such a picture, inhabitants of the white train would say that at a white time of 28 ticks grey rocket 1 was opposite white rocket 5 and its clock read 20 ticks. Inhabitants of the grey train, looking at the same picture, would say (equivalently) that at a grey time of 20 ticks white rocket 5 was opposite grey rocket 1 and its clock read 28 ticks. Note that the only difference in interpretation of the figures is that inhabitants of each train regard their clock as telling the correct time, and the clock on the other train as an interesting object whose reading, however, is not directly related to the time at which the picture was actually taken.

Suppose after the trains have gone past one another and large quantities of such information have been collected by the occupants of both trains, they return to the space station, and go off to separate rooms (a white room and a grey room) to compare notes on what pictures they took. What conclusions can they draw, if they act under the assumption that the different clocks on their own train were synchronized?

The first interesting thing to examine is any pair of pictures in each of which the same rocket appears. Figure 3, for example, shows two pictures in which grey rocket 0 appears, taken from parts (b) and (c) of Figure 1.

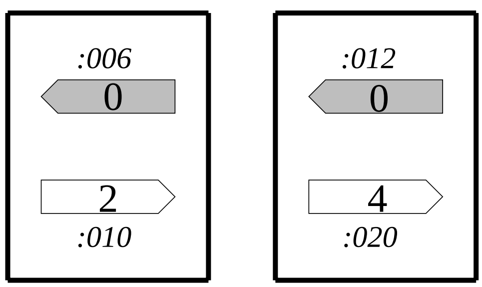


Figure 3

People on the white train will interpret these pictures as follows:

The most obvious thing they can read off from the two pictures is the speed of grey rocket 0, for in the first pictures it is opposite white rocket 2 at a time of 10 ticks, while in the second picture it is opposite white rocket 4 at a time of 20 ticks. So it went 2 rockets in 10 ticks, and is therefore travelling at a speed of $\frac{1}{5}$ rocket per tick.

The next thing the people on the white train can note from the two pictures is that at the white time of 10 ticks the clock on grey rocket 0 read 6 ticks, while at the later white time of 20 ticks, the clock on grey rocket 0 read 12 ticks. Therefore in the actual white time of 10 ticks that elapsed between the taking of the two pictures, the grey clock only advanced by 6 ticks. So it is running slowly by a factor of $\frac{3}{5}$.

Note that the validity of both these conclusions depends crucially on the assumption that the white clocks are synchronized, since the people from the white train are using the readings of two *different* clocks (one in white rocket 2 and the other in white rocket 4) to make their judgments about the times at which things happened.

Since Figure 1 is completely symmetric between grey and white, the grey-train people will evidently reach exactly the same conclusion about the white train and its clocks from any pair of pictures that show a single rocket from the white train — that the white train is moving at $\frac{1}{5}$ rocket per tick and its clocks are running slowly by a factor of $\frac{3}{5}$. This directly and straightforwardly reveals how a disagreement about whose clocks are correctly synchronized can lead to occupants of each of the two trains firmly convinced that it is the clocks on the other train that are running slowly. We, of course, taking the view in Figure 1 appropriate to the station frame, believe that both sets of clocks are running at exactly the *same* rate, and that *neither* set of clocks is correctly synchronized.

We now have both the speed v (1/5 rocket per tick) that the occupants of one train assign to the other train, as well as the slowing down factor s (3/5) they assign to the clocks on the other train. Anticipating that these ridiculously simple pairs of pictures extracted from the ridiculously simple full set of pictures in Figure 1 are going to mimic all the relativistic effects, we can note that an s of $\frac{3}{5}$ is associated with v/c of $\frac{4}{5}$ ($s = \sqrt{1 - v^2/c^2}$). Since $v = \frac{1}{5}$ rocket per tick, we should be on the alert for the speed of $\frac{1}{4}$ rocket per tick playing the role of an invariant velocity — the speed of light — in what follows.

The next interesting thing we can do is to examine any pair of pictures that were taken at the same time, according to one of the trains. Consider, for example, the two pictures taken at the grey time of 20 ticks, extracted from parts (c) and (d) of Figure 1, and shown in Figure 4. Because these pictures were taken at the same time, according to the occupants of the grey train, they immediately reveal to the grey-train people that the clocks on the white train are not synchronized. For at the grey time of 20 ticks the clock in white rocket 0 read 12 ticks, but that in white rocket 5 read 28 ticks. The white clocks disagree by 16 ticks and are 5 white rockets apart, so they are out of synchronization by $\frac{16}{5} = 3.2$ ticks per rocket.[3]

[3] You should not be surprised that this is different from the asynchronization of exactly 2 ticks per rocket evident in the station frame (Figure 1). A difference is to be expected, since people using the station frame know that the grey clocks are as badly out of synchronization as the white ones, and therefore the conclusions reached by the occupants of the grey train

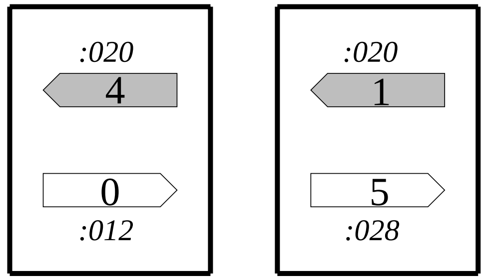


Figure 4

Furthermore people on the grey train can conclude that at a single moment of grey time — 20 ticks — five white rockets (rockets 4,3,2,1, and half each of rockets 5 and 0) stretched the same length as three grey rockets (rockets 2, 3, and half each of rockets 4 and 1) so the white rockets have shrunk by the same factor of $\frac{3}{5}$ as the white clocks are running slowly.

Notice that this amount of clock asynchronization, 3.2 ticks per rocket, is precisely what you would expect from the rule $T = Dv/c^2$ with the values we have found for v and c, $v = \frac{1}{5}$ rocket per tick and $c = \frac{1}{4}$ rocket per tick. For v/c^2 is then

$$\frac{\frac{1}{5}}{(\frac{1}{4})^2} = \frac{16}{5} = 3.2 \text{ ticks per rocket.} \tag{9.2}$$

This is exactly what we read off directly from Figure 4.

We can also understand how a different application of the Dv/c^2 rule fits in with the fact that people using the station frame know that the clock asynchronization on *both* trains is 2 ticks per rocket. If the invariant velocity is indeed $c = \frac{1}{4}$ rocket per tick, and if u is the speed of either train in the station frame, then the clock-asynchronization rule tells us that u/c^2 should be 2 ticks per rocket. This means that u, the speed of either train in the station frame, should be $\frac{1}{8}$ rocket per tick. But the speed of a train in the station frame is the same as the speed of the station in the train frame, and it is evident from parts (a) and (b) of Figure 1 that the speed of the station in the frame of either train is indeed $\frac{1}{8}$ rocket per tick.[4]

We can even check that these various speeds are consistent with the relativistic velocity addition law,

$$v_{wg} = \frac{v_{ws} + v_{sg}}{1 + (v_{ws}v_{sg}/c^2)}, \tag{9.3}$$

in such matters are quite unreliable.

4 For example the station is opposite rocket 0 at a time of 0 ticks, and opposite tocket 1 at a time of 8 ticks (in either the white or the grey train's frame).

where v_{wg} is the velocity of the white train in the frame of the grey train, v_{ws} is the velocity of the white train in the station frame, and v_{sg} is the velocity of the station in the frame of the grey train. We have $v_{ws} = v_{sg} = \frac{1}{8}$ rocket per tick and $c = \frac{1}{4}$ rocket per tick. When these numbers are put into (9.3) the result is indeed, $v_{wg} = \frac{1}{5}$ rocket per tick, so all the relativistic relations continue to hold.

I pause to emphasize again how very little has gone into the construction of Figure 1. The structure of part (a) of that figure is ridiculously simple. The only peculiar thing about it is the fact that the clocks do not all agree with each other. The way in which they disagree is evident. The rule for getting each of the other parts of the figure from the part above it is just to shift each train by one rocket in the direction it is going in, and advance every clock on each train by 6 ticks. Nothing elaborate has to be done to get relativity out of the figures. Once one introduces the asynchronized clocks on each train, all the other relativistic effects follow automatically.

The relativistic velocity addition law, for example, works for anything that moves between the two trains — not just the station itself. Consider, for example, an object that was between grey rocket 0 and white rocket 2 in part (b) of Figure 1 and between grey rocket 5 and white rocket 1 in part (d). It has been captured in the two pictures shown in Figure 5.

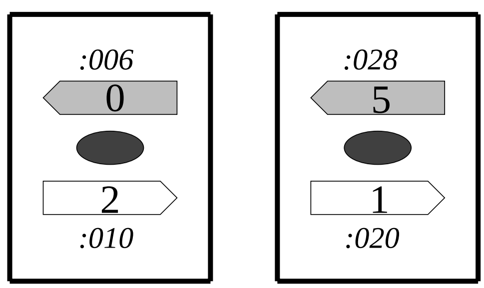


Figure 5

According to the grey train the object has gone 5 rockets to the right in 22 ticks, and according to the white train it has gone 1 rocket to the right in 10 ticks, so we have $v_{og} = \frac{5}{22}$ rocket per tick and $v_{ow} = \frac{1}{10}$ rocket per tick. We should have

$$v_{wg} = \frac{v_{wo} + v_{og}}{1 + (v_{wo} v_{og}/c^2)}, \tag{9.4}$$

which gives

$$v_{wg} = \frac{-\frac{1}{10} + \frac{5}{22}}{1 - (\frac{1}{10})(\frac{5}{22})/(\frac{1}{4})^2}, \tag{9.5}$$

which does indeed give $v_{wg} = \frac{1}{5}$ rocket per tick after all the arithmetic is carried out.

You can (and should) check for yourself that any other pair of pictures extracted from Figure 1 containing two moments in the history of a single object, yields values of v_{wo} and v_{og} that are consistent with the relativistic velocity addition law (9.4) and the facts that $v_{wg} = \frac{1}{5}$ rocket per tick and $c = \frac{1}{4}$ rocket per tick.

In particular, it is instructive to hunt around for a pair of photographs displaying two moments in the history of a single object moving at the special speed of $\frac{1}{4}$ rocket per tick. Figure 6 shows such a pair, taken from parts (c) and (d) of Figure 1. According to the grey train the object has moved 3 rockets to the right in a time of 12 ticks, so its velocity is $\frac{1}{4}$ rocket per tick. And according to the white train it has moved 1 rocket to the right in a time of 4 ticks, so its velocity is again $\frac{1}{4}$ rocket per tick. Such an object has the amusing ability to exploit the differences in clock synchronization on the two trains, in such a way that it can move along either train at the same speed, $\frac{1}{4}$ rocket per tick, provided the speed along a given train is timed by using the clocks carried by the rockets in that train, and provided those clocks are assumed to be synchronized.

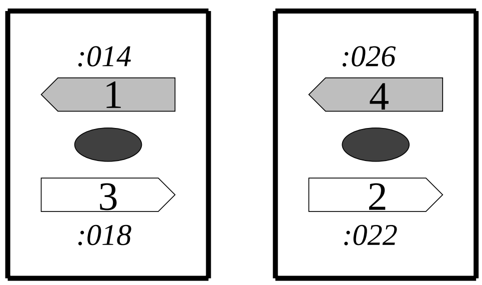


Figure 6

Figure 1 also provides us with a new insight into why motion faster than light is highly problematic. Figure 7 shows two pictures in the history of a hypothetical faster-than-light object taken from parts (c) and (d) of Figure 1. According to the grey train it has gone 6 rockets in 18 ticks, for a speed of $\frac{1}{3}$ rocket per tick, which exceeds the invariant velocity $c = \frac{1}{4}$ rocket per tick. People from the white train agree that the object goes faster than the invariant velocity, having gone 4 rockets in a mere 2 ticks, for a speed of 2 rockets per tick.[5]

But there is a very disturbing aspect to Figure 7. According to the grey train the picture on the left was taken 18 ticks before the one on the right. On the other hand

[5] You can check that even these superluminal velocities are consistent with the relativistic velocity addition law — but you have to be careful with the signs that indicate which way the object is moving in the frame of each train.

according to the white train, the picture on the left was taken 2 ticks *after* the one on the right. Occupants of the two trains disagree about the order in which the two pictures were taken! This is the kind of disagreement it is hard to tolerate. Suppose, for example, that the object were a burning candle. Its pictures would then clearly reveal the direction of time: the later the picture, the shorter the candle and the bigger the puddle of wax beneath it. Such a pair of pictures would clearly reveal to one of the groups that the clocks on its own train could not have been telling the correct time.

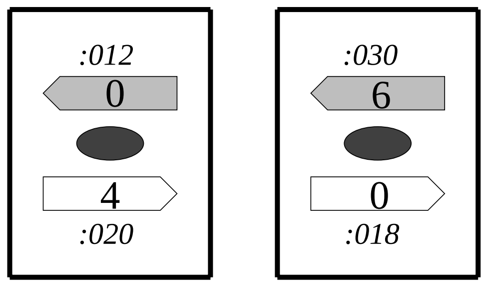


Figure 7

It turns out that this situation is quite general. If an object moves faster than light then there are always two frames of reference that disagree about the order in time of any pair of events in the history of the object.[6] Therefore if anything could move faster than light[7] it would have to be a sort of featureless blob, incapable of revealing, through its internal structure, any information about the direction of the flow of time. Burning candles, melting ice-cubes, rotting bananas, running-down batteries, aging people, and the like, cannot move faster than light.

Note, finally, that Figure 1 can also be used to demonstrate the invariance of the interval between two events. Take any pair of pictures whatever, and calculate

$$T^2 - D^2/c^2 = T^2 - (4D)^2 \tag{9.6}$$

(the 4 comes from the c^2, since c is $\frac{1}{4}$ of a rocket per tick), where T is the number of ticks between the events and D, the number of rockets. The answers will not depend on which frame you take to evaluate T and D. Consider, for example, Figure 8, which takes one event from part (b) and another from part (d) of Figure 1. According to the grey frame the two events are 22 ticks and 5 rockets apart, and $22^2 - (4\times5)^2 = 22^2 - 20^2 = 84$. According to the

[6] This is most easily demonstrated using the space-time diagrams that we will soon be developing.

[7] We have already seen from our application of the velocity addition law to rockets firing rockets firing rockets that the obvious way to bring this about does not work.

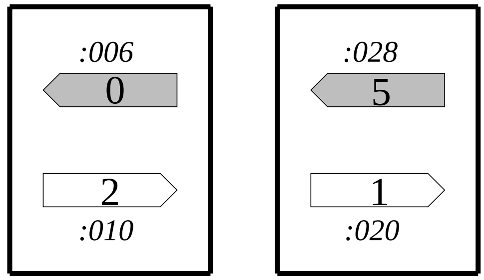


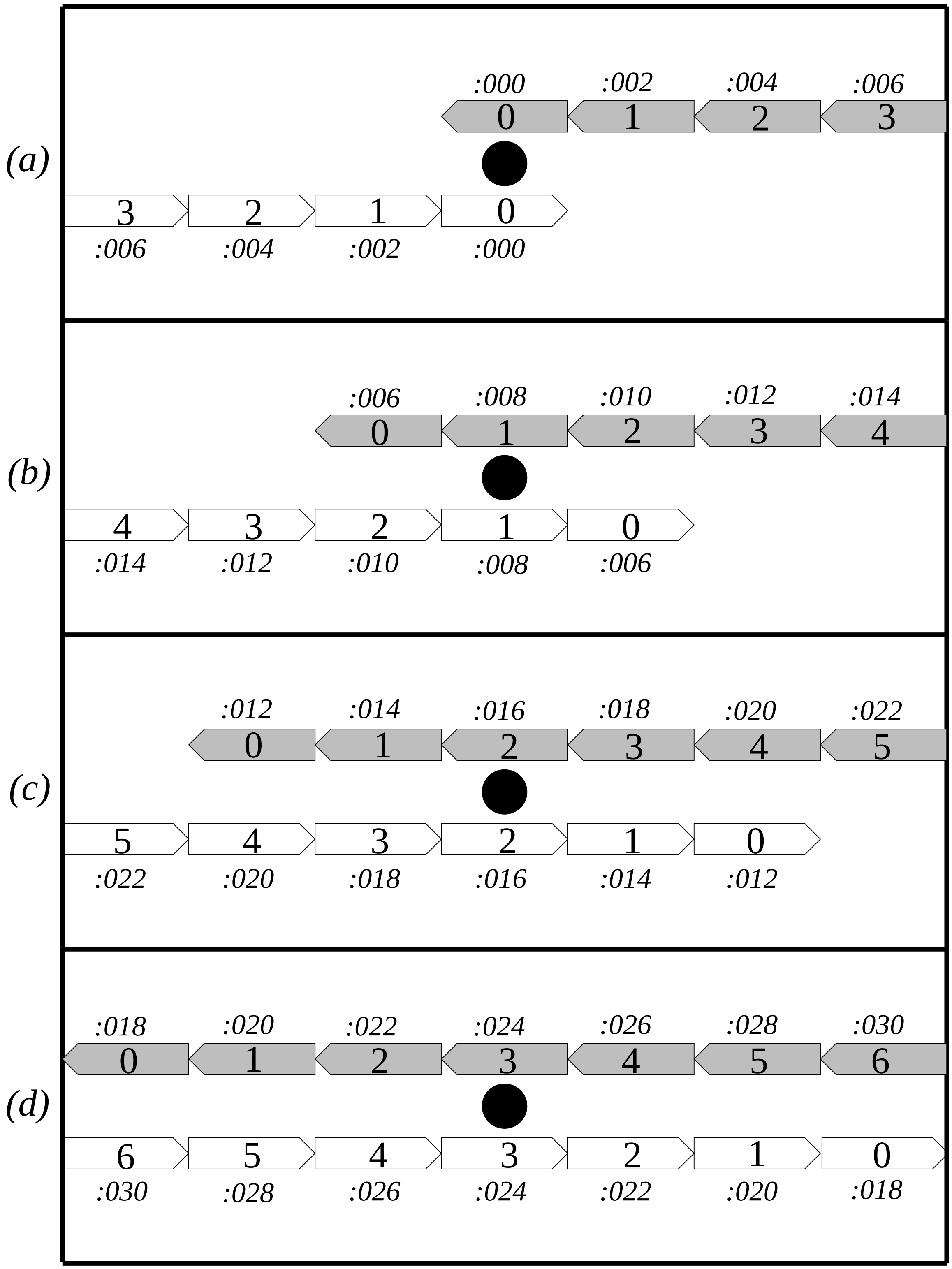
Figure 8

white frame the two events are 10 ticks and 1 rocket apart, and $10^2-(4\times1)^2 = 10^2-4^2 = 84$. This particular pair of events is time-like separated, since T^2-D^2/c^2 is positive, and indeed, an object present at both events would have a speed less than $\frac{1}{4}$ rocket per tick in either frame ($\frac{5}{22}$ rocket per tick in the grey frame and $\frac{1}{10}$ rocket per tick in the white frame. You can (and should) convince yourself that this works for any other pairs of events.

I have described all this as if the clocks on both trains were deliberately set out of synchronization by the people in the station frame and indeed, if that is how the trains and clocks were set up, and if the people on either train were under the impression that their clocks were actually synchronized, then they will interpret their photographs exactly as we have done.

What is special about the world we live in is this: If the people in the station frame should chose to do the experiment for trains moving with a speed of u feet per nanosecond, and should they choose the clock asynchronization to be exactly u nanoseconds of disagreement per foot of rocket, then to set up the clocks on both trains all they would have to do was to furnish people on each train with a highly accurate set of clocks, set the trains into motion, and instruct the people on each train to synchronize their clocks. Nature herself would automatically provide the discrepancy between the station-frame interpretation of the clocks, and the interpretation from within each train.

Figure 1


(a)
:000 :002 :004 :006
0 1 2 3
3 2 1 0
:006 :004 :002 :000
(b)
:006 :008 :010 :012 :014
0 1 2 3 4
4 3 2 1 0
:014 :012 :010 :008 :006
(c)
:012 :014 :016 :018 :020 :022
0 1 2 3 4 5
5 4 3 2 1 0
:022 :020 :018 :016 :014 :012
(d)
:018 :020 :022 :024 :026 :028 :030
0 1 2 3 4 5 6
6 5 4 3 2 1 0
:030 :028 :026 :024 :022 :020 :018