

## 7. What You See When You Look at a Moving Clock

We have established that in any inertial frame of reference a clock that moves with speed  $v$  runs slowly compared with stationary clocks. The slowing down factor is given by<sup>1</sup>

$$s = \sqrt{1 - v^2/c^2}. \quad (7.1)$$

One has the feeling that this is just some kind of trick — a conclusion based on playing sneaky intellectual games with the concept of simultaneity. If you actually *looked* at a moving clock would you actually *see* it running slowly?

The answer is that what you see depends on whether the clock is moving towards you or away from you. If it moves towards you, you actually see it running fast! If it moves away from you, you do indeed see it running slowly, but considerably more slowly than the (slow) rate at which it actually is running. This disparity between how fast a clock runs and how fast you *see* it run is a simple consequence of the fact that you do not *see* a clock reading a particular number until light that has left the clock at the moment it displays that number travels from the clock to your eyes.

If the clock is standing still this delay doesn't matter at all, because the extra time between the clock flashing each new number<sup>2</sup> and the light actually reaching you from each new flash is the same for each number. So although there is a delay before you see each flash, you receive the flashes at the same rate the clock is emitting them, and therefore you see the clock running at its actual rate (though a little behind what it actually is reading at the moment you see it because of the delay).

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<sup>1</sup> Here is a possible (silly) source of confusion. Whether you should multiply or divide by the slowing down factor depends on what question you want to answer. The things to keep in mind are (a) that moving clocks run *slowly* and (b) that the slowing-down factor  $s$  is *less* than 1. So if the question is “How much does the reading on a moving clock advance in a time  $T$ ?” the answer is that it advances by  $sT$ , since the moving clock runs slowly and multiplying  $T$  by  $s$  gives a number less than  $T$ . But if the question is how much time does it take for the reading on a moving clock to advance by  $T$ , the answer is  $T/s$ , since it takes a clock running slowly more time than  $T$ , and dividing  $T$  by a number  $s$  less than 1 produces a bigger number. Memorizing a formula doesn't work. Thinking does.

<sup>2</sup> It might be helpful to think of the clock as a digital clock that signals its reading by flashing a number. Of course even when you are looking at an ordinary mechanical clock, the only reason you can see it is that light has bounced off its hands and then travelled to your eyes at the speed of light.

But if the clock is moving away from you the light from each successive flash has further to go before it reaches you, so you see the clock running even more slowly than it actually is running. On the other hand if the clock is moving towards you, the light from each successive flash has less distance to cover, so you see the clock running faster than it actually is running. It turns out that when the clock moves towards you this effect is more important than the fact that the clock is running slowly, so you *see* it running fast. Conversely if the clock moves away from you, you *see* it running considerably more slowly than it actually is running.

It is not hard to construct a quantitative measure of this effect, which is called the relativistic *Doppler effect*. In fact it is possible to do so without knowing the actual value (7.1) of the slowing down factor  $s$ , in a manner that establishes what that value is, quite independently of the argument we gave in Part 6, based on the  $T = vD/c^2$  rule for simultaneous events. The argument goes like this:<sup>3</sup>

Take a clock that flashes a new number every  $T$  seconds in its proper frame. Let  $f_t T$  and  $f_a T$  be the number of seconds between the flashes that reach you when the clock moves toward ( $t$ ) or away ( $a$ ) from you with speed  $v$ . We can deduce the values of  $f_t$ ,  $f_a$ , the speeding up or slowing down factors for what you *see*, as well as the value of  $s$ , the slowing down factor for what the clock is actually doing, by the following line of thought:<sup>4</sup> since the moving clock runs slowly it only flashes a new number every  $T/s$  seconds. During that time it gets a distance  $v(T/s)$  further away (or closer to) you, so the light from each successive flash takes a time  $v(T/s)/c$  more (or less) to get to you. Consequently the time between light from the flashes reaching you (and therefore the time between your *seeing* successive flashes) is

$$f_a T = T/s + v(T/s)/c = (T/s)(1 + v/c) \quad (7.2)$$

if the clock moves away from you. And it is

$$f_t T = T/s - v(T/s)/c = (T/s)(1 - v/c) \quad (7.3)$$

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<sup>3</sup> The argument that follows provides a derivation of the fact that  $s = \sqrt{1 - v^2/c^2}$  that is independent of the analysis in Part 6. It also provides a derivation of the relativistic velocity addition law independent of the one given in Part 4. From a strictly logical point of view there is no need for independent arguments leading back to conclusions we have already established, but it is important to see that the same conclusions emerge from quite different lines of thought.

<sup>4</sup> The talk that follows anticipates the fact (which we already know, but are about to rederive) that  $s$  is less than one. But the argument (with a few words suitably changed) would work just as well if  $s$  were greater than one.

if the clock moves towards you. Therefore

$$f_a = (1/s)(1 + v/c) \tag{7.4}$$

and

$$f_t = (1/s)(1 - v/c). \tag{7.5}$$

Since we already know the value of the slowing down factor  $s$ , we are finished. Even, however, if we didn't know the value of  $s$ , we are now in a position to figure it out from the following simple but clever idea:

Suppose Alice and Bob are stationary in the same frame of reference at different places, and Bob holds a clock that Alice watches. Suppose Bob's clock flashes every second in its proper frame. Since the clock is stationary with respect to Alice, every flash takes the same time to reach her, and Alice sees a flash every second. Now suppose that Carol moves from Bob to Alice at speed  $v$ . Each time Carol sees a new number appear on Bob's clock, she reinforces it with a number of her own. She can do this automatically by setting a clock moving with her to flash a new number at the same rate that she sees Bob's numbers. Since she moves *away* from Bob with speed  $v$  she sees a flash from Bob's clock every  $f_a$  seconds. She therefore adjusts her flasher to emit a number every  $T = f_a$  seconds. Since Carol and her flasher move toward Alice at speed  $v$  Alice sees Carol's flasher flashing every  $f_t T = f_t f_a$  seconds. But since Carol's flashes arrive together with Bob's, and Alice sees one of Bob's flashes every second, Alice must also see one of Carol's flashes every second. *The net effect of Carol's seeing Bob's clock flash slowly and Alice seeing Carol's clock flash fast must cancel precisely:*

$$f_t f_a = 1. \tag{7.6}$$

When we combine (7.6) with (7.4) and (7.5), we learn everything of interest. Note first that (7.4) and (7.5) together require that

$$f_t f_a = (1/s)^2 (1 + v/c)(1 - v/c) = (1/s)^2 (1 - v^2/c^2). \tag{7.7}$$

In view of (7.6) this immediately gives us an independent way of arriving at the form (7.1) of the slowing down factor  $s$ . On the other hand (7.4) and (7.5) also tell us that

$$f_t/f_a = \frac{1 - v/c}{1 + v/c}. \tag{7.8}$$

Combining this with (7.6), which tells us that  $1/f_a = f_t$ , we immediately learn that

$$f_t = \sqrt{\frac{1 - v/c}{1 + v/c}}, \tag{7.9}$$

and therefore  $f_a$  (which is  $1/f_t$ ) is given by

$$f_a = \sqrt{\frac{1 + v/c}{1 - v/c}}. \quad (7.10)$$

Suppose, for example, that  $v = \frac{3}{5}c$ , so the slowing down factor is  $\sqrt{1 - (\frac{3}{5})^2} = \frac{4}{5}$ . This tells us that clock moving at 60% of the speed of light takes  $\frac{5}{4} = 1.25$  seconds to flash each second — it runs at  $\frac{4}{5} = 80\%$  of its normal rate. But

$$\sqrt{\frac{1 - \frac{3}{5}}{1 + \frac{3}{5}}} = \frac{1}{2}, \quad (7.11)$$

so if the clock is moving toward you you see it flash a new second every half-second — i.e. you see it running at *twice* its normal rate. If it moves away from you you see it flash a new second every two seconds — i.e. you see it running at *half* its normal rate. If  $v = \frac{4}{5}c$  the slowing down factor drops to  $\frac{3}{5}$ . But according to (7.9) and (7.10) the rate at which you *see* the clock flash differs from the rate at which it runs in its proper frame by a factor of 3.

The form (7.10) for the rate at which you see a clock ticking as it moves away from you is of great importance in cosmology. Distant galaxies moving away from us contain clocks in the form of atoms — identical to those on earth — whose characteristic internal vibrations lead to the emission of light that vibrates at the rate the atom vibrates in its proper frame. The rate at which we see the light that reaches us vibrating is slower than the rate at which the atoms vibrate in their proper frame for just the reasons we have been discussing. This slowing of the perceived vibration rate is called<sup>5</sup> the “red shift” (because red light vibrates more slowly than the other colors of the visible spectrum). The amount by which the vibration rate is reduced is given precisely by the relativistic result (7.10). One can therefore turn things around and use the amount of the red shift to deduce the speed at which the distant galaxy moves away from us. A red shift by a factor of 2, for example means (see Eq, (7.11)) that a galaxy is moving away at 3/5 of the speed of light.

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<sup>5</sup> It is an optical version of the Doppler effect, in which the pitch of a note is higher or lower depending on whether the instrument sounding the note is moving towards or away from you. In this case the relativistic slowing down of the vibration in the instrument is so small as to be inconsequential, but the fact that the instrument moves toward or away from you while vibrating continues to be important. (And, of course, the speed that plays the role of  $c$  is now the speed of sound with respect to the air.)

## Alternative derivation of the relativistic velocity addition law

By a modest generalization of the argument leading to (7.6) we can also extract the relativistic velocity addition law in an way that is entirely different from what we did in Part 4.

Suppose Bob and Charles move to the right away from Alice at speeds  $v$  and  $w$  in Alice's frame of reference, and Charles moves to the right away from Bob at speed  $u$  in Bob's frame. If Alice has a clock that flashes once a second, then Charles will receive a flash from her every  $\sqrt{\frac{1+w/c}{1-w/c}}$  seconds, while Bob will receive a flash from Alice every  $\sqrt{\frac{1+v/c}{1-v/c}}$  seconds. So if Bob reinforces the flashes from Alice by setting his own clock to flash at the same rate he receives Alice's flashes, then Charles will receive the reinforcing flashes from Bob every  $\sqrt{\frac{1+u/c}{1-u/c}}\sqrt{\frac{1+v/c}{1-v/c}}$  seconds.<sup>6</sup> Since this must coincide with the rate at which he directly receives the flashes from Alice, we must have

$$\sqrt{\frac{1+w/c}{1-w/c}} = \sqrt{\frac{1+u/c}{1-u/c}} \sqrt{\frac{1+v/c}{1-v/c}}. \quad (7.12)$$

But (7.12) is entirely equivalent<sup>7</sup> to the velocity addition law written in its multiplicative form,

$$\frac{c-w}{c+w} = \left(\frac{c-u}{c+u}\right)\left(\frac{c-v}{c+v}\right). \quad (7.13)$$

I conclude with a puzzle: The above argumentation leads to the relativistic slowing-down factor (7.1) from nothing more than some simple considerations on how fast a clock looks to be running as it moves towards or away from you. The phenomenon of relativistic slowing down, however, depends crucially on the principle of the constancy of the velocity of light, which I never appealed to in the course of the above argument. What's going on here?<sup>8</sup>

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<sup>6</sup> The factor on the left gives the number of seconds between flashes of Bob's clock in Bob's frame — namely the number of seconds between the flashes he receives from Alice. The factor on the right changes this to the number of seconds between the flashes Charles receives from Bob's flashing clock, in view of the fact that Charles is moving away from Bob with speed  $v$ .

<sup>7</sup> Convince yourself of this equivalence. Don't just passively take my word for it. It should be "obvious" but if your algebraic skills are a little rusty, you might have to think about it just a bit.

<sup>8</sup> If you like puzzles put down the page and think, before turning over the page.

The answer is that although I never explicitly invoked that principle, I sneaked it into the argument in a rather subtle way. We derived the relations (7.9) and (7.10) in a frame of reference in which the person watching the clock was stationary, and the clock was moving. When we applied those relations to get (7.6) we ought to have used the form for  $f_a$  appropriate in Carol's frame and the form for  $f_t$  appropriate in Alice's. The appearance of one and the same value for  $c$  in both Carol's  $f_a$  and Alice's  $f_t$  only makes sense if the speed of all those flashes of light is the same in both Carol's frame and Alice's.