6. Slowing down of Moving Clocks; Shrinking of Moving Objects

In Chapter 5 we concluded that if two clocks are synchronized and separated by a distance D in a frame in which they are at rest, then in a frame in which they move with speed v along the line joining them, they are not synchronized: the clock in front is behind the clock in the rear by an amount T given by[1]

$$T = Dv/c^2. \tag{6.1}$$

By exploring the consequences of this fact for two appropriately chosen clocks synchronized in the train-frame (used by Alice) and two other appropriately chosen clocks synchronized in the track-frame (used by Bob), we now deduce that moving clocks must run slowly and that moving trains (or moving tracks) must shrink along the direction of their motion. We can also deduce the precise amount by which the clocks slow down and the trains or tracks shrink.

Let the proper length of the train (i.e. the length of the train in the train (Alice's) frame) be L_A. Let a clock be attached to each end of the train, as shown on the right half of Figure 1[2] on page 6, which depicts things as they are described in the train frame. Both clocks are synchronized in the train frame, so both read the same time: 0.

Because the clocks are synchronized in the train frame, they are not synchronized in the track frame. This is shown on the left half of Figure 1. Note that because the train clocks are *not* synchronized in the track frame, it requires *two* track frame pictures taken at two different track frame times, to depict both of them reading 0. In the upper left picture the clock at the rear of the train reads zero, and the clock in the front is behind the clock in the rear, reading a negative time $-T_A$. In the lower left picture the clock in the

[1] A neat way to say this is that the clock in front is behind the clock in the rear by v nanoseconds per foot of proper-frame separation, where v is the speed of the clocks in f/ns. Since v is necessarily less than 1 f/ns, this isn't an enormous effect. It's less than a microsecond per 1000 feet, and substantially less, if v is substantially less than the speed of light. If v is the speed of sound (a foot per millisecond) it's only a millionth of a microsecond per 1000 feet, or a nanosecond per million feet — half a nanosecond per thousand miles.

[2] As usual, to make sense of what I'm saying you must refer to the figure as you read the text that follows, and to read the caption of the figure before you read any further in the text.

front of the train has advanced from $-T_A$ to 0, while the clock in the rear has advanced by the same amount[3] from 0 to $+T_A$. The track-frame time between the two pictures is the time T_B that the two clocks attached to the tracks[4] have advanced.

The quantitative rule (6.1) tells us that the amount T_A by which the two train-frame clocks differ in the track frame is related to the train-frame distance L_A between then by

$$T_A = L_A v/c^2. \tag{6.2}$$

By the same token the amount T_B by which the two track-frame clocks differ in the train frame is related to the track-frame distance D_B between them by

$$T_B = D_B v/c^2. \tag{6.3}$$

From this information, together with a few other simple features of Figure 1, we can deduce that moving clocks must run slowly, that moving trains or tracks must shrink along the direction of their motion. We can even deduce the exact amount of the slowing-down and shrinking.

The slowing-down[5] factor for moving clocks is given by T_A/T_B. To see this look at the two track frame pictures on the left of Figure 1. Between the two pictures both track frame clocks advance by a time T_B, while both train-frame clocks advance by a time T_A. Since the track-frame clocks give correct time in the track frame, T_A is the time a train-frame clock advances in a track-frame time T_B. So the ratio T_A/T_B does indeed measure how much the moving train-frame clocks run slowly in the track frame.[6]

In the same way, the shrinking factor[7] for a moving object is given by the ratio L_B/L_A of the track-frame length L_B of the train to its length L_A (its proper length) in the frame in which it is at rest. It is also given by the ratio D_A/D_B of the train-frame length of the moving track between the two track-frame clocks and the proper length D_B of that same stretch of track:[8]

$$L_B/L_A = D_A/D_B. \tag{6.4}$$

[3] They have advanced by the same amount because they are identical clocks moving at the same speed.

[4] These two clocks are synchronized in the track frame, as is evident from the pictures on the left.

[5] If it turned out to be a number bigger than 1 it would be a speeding-up factor, but I'm anticipating the fact that it turns out to be less than 1.

[6] For example if the ratio T_A/T_B were 0.8 then a train-frame clock would gain only 0.8 seconds in a track-frame second.

[7] If it turned out to be a number bigger than 1 it would be a stretching factor, but I'm again anticipating the fact that it turns out to be less than 1.

[8] At the risk of being boorish, may I again remind you to check these assertions against Figure 1.

To deduce the form of the shrinking and slowing-down factors we need note only two other things:

(1) It is evident from the train-frame picture on the right of Figure 1 that the train-frame length D_A of (moving) track connecting the two track frame clocks is equal to the train-frame (proper) length L_A of the train:

$$L_A = D_A. \tag{6.5}$$

It is crucial for the validity of this conclusion that the train-frame synchronized clocks at the two ends of the train both read the same time (0), so the picture shows a single moment of train-frame time. Were the figure, on the contrary, a composite stitched together from fragments taken at different moments of train-frame time, then different parts of the tracks, which move in the train frame, would be pictured at the places they occupied at different moments of time, and we could conclude nothing about the train-frame length of the piece of track between the clocks.

(2) The track-frame pictures on the left give a relation between L_B and D_B only slightly more complicated than (6.5). According to those pictures D_B is the track-frame distance between the left end of the train at track-frame time 0 and the right end at track-frame time T_B. This distance is given by the track-frame length L_B of the train *plus* the distance the train moves between the two pictures. Since the track-frame time between the two pictures is T_B and the train moves with speed v, that additional distance is vT_B, so we have

$$D_B = L_B + vT_B. \tag{6.6}$$

Everything we need to know follows from the relations (6.2)-(6.6). To begin with, we can conclude immediately from the relations (2), (3), and (5) that the slowing-down factor for moving clocks must be *the same* as the shrinking factor for moving sticks.[9] For (6.2) and (6.3) tell us[10] that $T_A/T_B = L_A/D_B$, while (6.5) tells us that $L_A = D_A$. Consequently

$$T_A/T_B = D_A/D_B. \tag{6.7}$$

The left side of this relation is the slowing-down factor for moving clocks; the right side is the shrinking factor for moving objects. Calling this factor s, we can write

$$T_A = sT_B, \quad D_A = sD_B, \quad \text{and also} \quad L_B = sL_A \tag{6.8}$$

[9] Another very simple independent demonstration of this fact that makes no use of the rule (6.1) for synchronized clocks is given in the Appendix at the end of this essay.

[10] At the risk of being really irritating, may I remind you that when I make an assertion like this you should look back at the equations I cite to confirm that they really do tell us what I claim they do.

(where the last of these follows from (6.4).)

To find the actual value of the shrinking (slowing-down) factor s, note that if we combine (6.6) with (6.3), we find that $D_B = L_B + v^2 D_B/c^2$, which tells us that

$$L_B = D_B(1 - v^2/c^2). \tag{6.9}$$

But (6.8) tells us that $L_B = sL_A$, (6.5) tells us that $L_A = D_A$, and (6.8) tells us that $D_A = sD_B$. Putting these together tells us that $L_B = s^2 D_B$, and therefore (6.9) tells us that

$$s^2 D_B = D_B(1 - v^2/c^2). \tag{6.10}$$

Consequently the shrinking factor (or slowing-down factor) is[11]

$$s = \sqrt{1 - v^2/c^2}. \tag{6.11}$$

This shrinking of moving objects along the direction of their motion is called the Fitzgerald contraction, in honor of the otherwise little-known Irish physicist who first suggested it. It is also called the Lorentz–Fitzgerald contraction, in honor of the great Dutch physicist H. A. Lorentz, who had the same idea at about the same time. Often it is just called the Lorentz contraction, a manifestation of the unfortunate Matthew effect.[12]

The slowing down of moving clocks is often referred to by the deplorable term "time dilation." It is deplorable because it suggests in some vague way that "time itself" (whatever that might be) is expanding. While the notion that time stretches out for a moving clock has a certain intuitive appeal, it is important to recognize that what we are actually talking about has nothing to do with any overarching concept of "time". It is simply a relation between two sets of clocks.[13] If one set of clocks is considered to be stationary,

[11] Note that this is the square root of a number less than 1, so s is indeed less than 1 and is indeed a shrinking (not stretching) or slowing-down (not speeding-up) factor. Note also that if v exceeds c then (6.10) tells us that s^2 is a negative number, which makes no sense. Indeed, the analysis used in the lecture notes on simultaneity and clock synchronization only makes sense if the train is moving at a speed v less than the speed of light c. (In a frame in which the train moves faster than the speed of light the photon from the middle of the train will *never* reach the front of the train.) Considerations like these provide further hints that the speed of light is an upper limit to how fast anything can be moving in any inertial frame of reference.

[12] "To him that hath shall be given; from him that hath not shall be taken even that which he hath."

[13] While it is commonly believed that there is something called time that is measured by clocks, I would argue that the concept of "time" is nothing more than a convenient (though potentially treacherous) device for summarizing compactly all the relationships holding between different clocks. Not all my physicist colleagues agree with me about this.

synchronized, and running at the correct rate, then a second set, considered to be moving, will be found to be both asynchronized[14] and running slowly, according to the first set. But if we consider the second set to be stationary, synchronized an running at the correct rate, then the first set will be found to be asynchronized and running slowly according to the second.

In both cases — the shrinking of moving sticks or the slowing down of moving clocks — one is inclined to be deeply suspicious of these conclusions. How can Alice maintain that Bob's clocks are running slowly, and Bob maintain that Alice's clocks are running slowly, when they are both talking about the same set of clocks? If Alice maintains that Bob's clocks are running slowly, shouldn't Bob necessarily maintain that Alice's clocks are running *fast*? Similarly for sticks, lined up along their direction of relative motion. If Alice maintains that Bob's moving sticks have shrunk compared with her stationary sticks, then shouldn't Bob have to maintain that Alice's moving sticks have *stretched* compared with his stationary sticks?

To succumb to this suspicion is to forget that Alice and Bob also disagree on whether two events in different places happen at the same time or, equivalently, on whether two clocks in different places are synchronized. Because of this disagreement each of them maintains that the other has determined the rate of a moving clock or the length of a moving stick *incorrectly*. For to measure the length of a *moving* stick one must determine where its two ends are *at the same time*[15] and this requires a judgment about whether spatially separated events at the two ends of the stick are or are not simultaneous. And to compare how fast a moving clock is running compared with stationary clocks, it is necessary to compare at least two of the readings of the moving clock with the readings of nearby stationary clocks; but since the moving clock moves, this requires one to use two correctly *synchronized* stationary clocks in two different places.

There is thus nothing inconsistent in Alice and Bob each saying that the other's clocks are running slowly and each saying that the other's sticks have shrunk. Each can point to a flaw — a failure to use properly synchronized clocks — in the procedure that the other uses to make such determinations. This is not, however, to say that the phenomena of time dilation and length contraction are mere conventions about how we use language to describe the behavior of clocks and measuring sticks. As we shall see, they can have

[14] Even though they are considered to be synchronized in the frame in which they are both stationary.

[15] If you don't get the locations of the two ends at the same time, then the stick will have moved between your two determinations of where its ends are, and you won't be getting its length right.

quite striking consequences. It's just that the *explanations* given for those consequences can differ dramatically from one frame of reference to another.

A simple manifestation of this behavior, which has actually been observed, is provided by the behavior of unstable elementary particles. These have a characteristic lifetime τ. If you have a group of such particles, at rest or moving slowly, about half of them disintegrate within a time τ. Their collective statistical behavior therefore provides a kind of clock. Furthermore, in contrast to watches or alarm clocks, it is possible using currently available technology to get the moving at speeds u very close to the speed of light.[16] When a group of such particles is moving along a track at a speed close to the speed of light c, most of the particles in the group manage to travel without disintegrating over distances of track very much greater than the typical distance $u\tau$ one would expect them to be able to cover if their survive rate were unaffected by their motion. They can go much further because their "internal clocks" that govern when they decay are running much more slowly in the frame in which they rush along at speeds close to c. This is a real effect, and it plays a crucial role in the operation of such particle accelerators.

But how can this behavior be reconciled with the fact that in the frame moving with the particles, their internal clocks are running at the normal rate, and only about half of them can survive for a time τ? The reconciliation comes from the fact that in that frame, the track along which the particles move is rushing by at close to the speed of light. All distances along the track are therefore reduced by the shrinking factor, and much more of the track can go past the particles in the time τ than could have gone past if the track had not shrunk.

So both frames agree that half the particles are able to cover a greater length $u\tau/s$ of track, where s is the shrinking or slowing-down factor for things moving with the velocity of the particles, which can be very small if the velocity is close to c. In the track frame this is because a typical particle survives for a time τ/s which is much longer than the time τ it would survive if it were stationary. But in the rest frame of the particles it is because the length of the track has shrunk by a faster s so the length of moving track that can get past the particle in the time τ is augmented by the factor $1/s$. The explanatory stories differ, but the resulting behavior is the same.

This effect was observed in the behavior of particles called μ-mesons well before the age of enormous particle accelerators. These are produced by cosmic rays in the upper atmosphere. When at rest they have a lifetime of about 2 microseconds, so if their internal clocks ran at a rate independent of their speed, even if they travel at the speed of light

[16] This is done, for example, in the synchrotron at Wilson Lab on the Cornell campus.

about half of them would be gone after they have travelled 2000 feet. Yet about half the μ-mesons produced in the upper atmosphere (about 100,000 feet up) manage to make it down to the ground. This is because they travel at speeds so close to the speed of light that the slowing down factor is $s = 1/50$, and they can survive for 50 times as long as they can when stationary. In the frame of the μ-mesons, of course, their lifetime remains 2 microseconds, but half of them still make it down to the ground because the earth is rushing up at them so fast that the height of the atmospher contracts by a factor of $1/s = 50$, from 100,000 feet to 2,000 feet.

It is important to note that although there are substantial disagreements between Alice's train-frame picture of events and Bob's track-frame picture, whenever the two pictures are narrowed down to describe only things that happen both in the same place and at the same time, both restricted pictures agree. This is illustrated in Figure 2. This is quite a general state of affairs. All frames of reference will agree in their description of space-time coincidences — events which happen both at the same time and in the same place. Differences of opinion only arise when it comes to "stitching" together such events to tell a more elaborate story of what things are like everywhere at a given time. The disagreements arise because "at a given time" means different things in different frames. When this is fully taken into account, the disagreements are revealed as merely different conventional ways of describing the same phenomena.

Here is a concyise summary of the basic facts about clocks and measuring sticks:

Rule for synchronized clocks

If two clocks are stationary, synchronized, and separated by a distance D in Alice's frame, then in a second frame, Bob's, in which they are moving with speed v along the line joining them, the clock in front is behind the clock in the rear by[17]

[17] This is equivalent to the rule for simultaneous events. For when Bob says that the clock in front is behind the clock in the rear by T, he means that the event consisting of the clock in front reading 0 (for example) is *simultaneous* with the event consisting of the clock in the rear reading T. But since the clocks are *synchronized* according to Alice, these two events, although simultaneous for Bob, are *not* simultaneous for her. Indeed, since the clocks tell correct time for Alice, in her frame the time between the two events is the difference in the clock readings: $T = Dv/c^2$. Furthermore since the clocks don't move in Alice's frame, D is the distance between two events in the history of the two clocks, even if the events are not simultaneous. Consequently Alice has an example of two events a distance D apart in her frame, that are simultaneous in Bob's frame, but are a time $T = Dv/c^2$ apart in her frame.

$$T = Dv/c^2. \tag{1}$$

Rule for shrinking of moving sticks or slowing down of moving clocks

The shrinking (or slowing-down) factor s associated with a speed v is given by

$$s = \sqrt{1 - v^2/c^2}. \tag{2}$$

A clock moving with speed v runs slowly by a factor s. So in T seconds according to stationary clocks, the moving clock only advances by sT seconds. Or, putting it another way, according to stationary clocks it takes t/s seconds for the reading of the moving clock to advance by t seconds.

A stick moving along its own direction with speed v shrinks by a factor s. So if the stick has proper length L, then when it moves with speed v its length is only sL.

Appendix

There is a simple reason why the slowing down factor s for moving clocks must be the same as the shrinking factor s for moving sticks. This explanation makes no use of the "Dv/c^2" rule for simultaneous events. The reason is that if the slowing down factor and shrinking factors were different, then the behavior of the moving μ-mesons would come out differently depending on whether you calculated it in the rest frame of the mesons or the rest frame of the tracks. Here is a slightly more abstract way to make the same point:

Suppose (see Figure 3) we have a stick of proper length L along which a clock moves to the right with speed v. Let the clock read 0 as it passes the left end of the stick and T as it passes the right end. This is depicted in the stick-frame in the two pictures on the left of Figure 3. The same stick and clock are depicted in the clock-frame in the two pictures on the right of Figure 3.

If s_1 is the shrinking factor for moving sticks, then in the clock frame we have a stick of length $s_1 L$ moving with speed v to the left. The time it takes between the left end of the stick passing the clock and the right end passing it, is just the time it takes the left end to go a distance $s_1 L$ to the left of the clock. Since the stick moves with speed v, that time is $T = s_1 L/v$. Since the clock tells correct time when it is stationary and reads 0 when the left end of the stick passes it, $T = s_1 L/v$ must be what it reads when the right end of the stick passes it.

In the stick frame, on the other hand, the time it takes the clock to go from the left to the right end of the stick is L/v, the unshrunken length L of the stick divided by the speed v of the clock. But since the clock runs slowly in the stick frame, during this time the reading of the clock advances only by $T = s_2 L/v$, where s_2 is the slowing-down factor for moving clocks.

Since both frames must agree on the time T showing on the clock as it passes the right end of the stick, the shrinking factor s_1 must be the same as the slowing-down factor s_2.

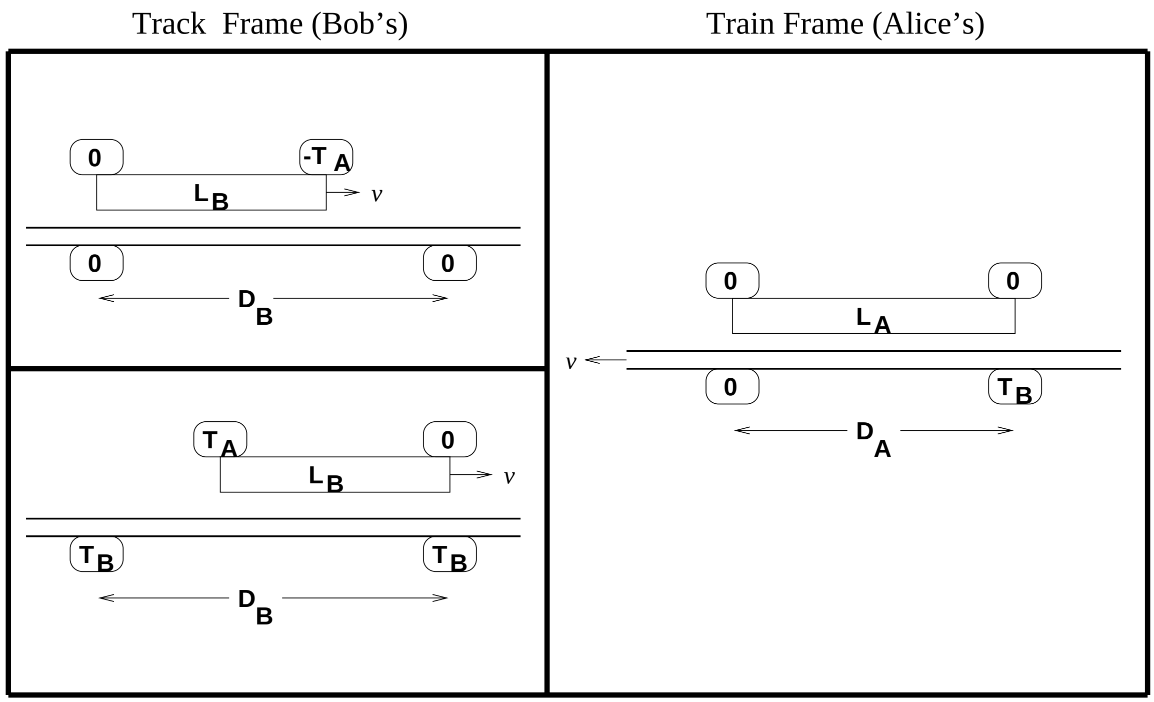

Figure 1. The figure shows three different pictures (in the three boxes bounded by heavy black lines.) Each picture shows four clocks, a train, and a track. The train is the long rectangle. Two of the clocks are attached to it, one at the front, the other at the rear (the small rounded rectangles just above the front and rear of the train.) The tracks are the two long parallel lines below the train. The other two clocks are attached to the tracks (the two small rounded rectangles shown below the tracks.) The clocks attached to the train are synchronized in the train frame; those attached to the tracks are synchronized in the track frame. The time shown by a clock is indicated by the symbol inside it.

The picture on the right depicts a single moment of time in the train frame. Both train clocks read the same time 0. The track and its attached clocks move to the left with speed v. The track clocks are not synchronized in the train frame: the clock in the front is behind the clock in the rear by a time T_B. The length of the train in the train frame is its proper length, L_A. The two clocks attached to the track are directly opposite the two clocks attached to the two ends of the train. It is evident from the figure that L_A is the same as D_A, the train frame length of the segment of (moving) track that stretches between either pair of clocks.

The two pictures on the left depict two different moments of time in the track frame. The first picture takes place when both track frame clocks read 0; the second takes place when both read T_B. The train and its attached clocks move to the right with speed v. Note that the train frame clocks are not synchronized in the track frame: the clock in front is behind the clock in the rear by a time T_A. The distance between the clock attached to the tracks in the track frame is the proper length D_B of the segment of track between them. The length of the train in the track frame is L_B.

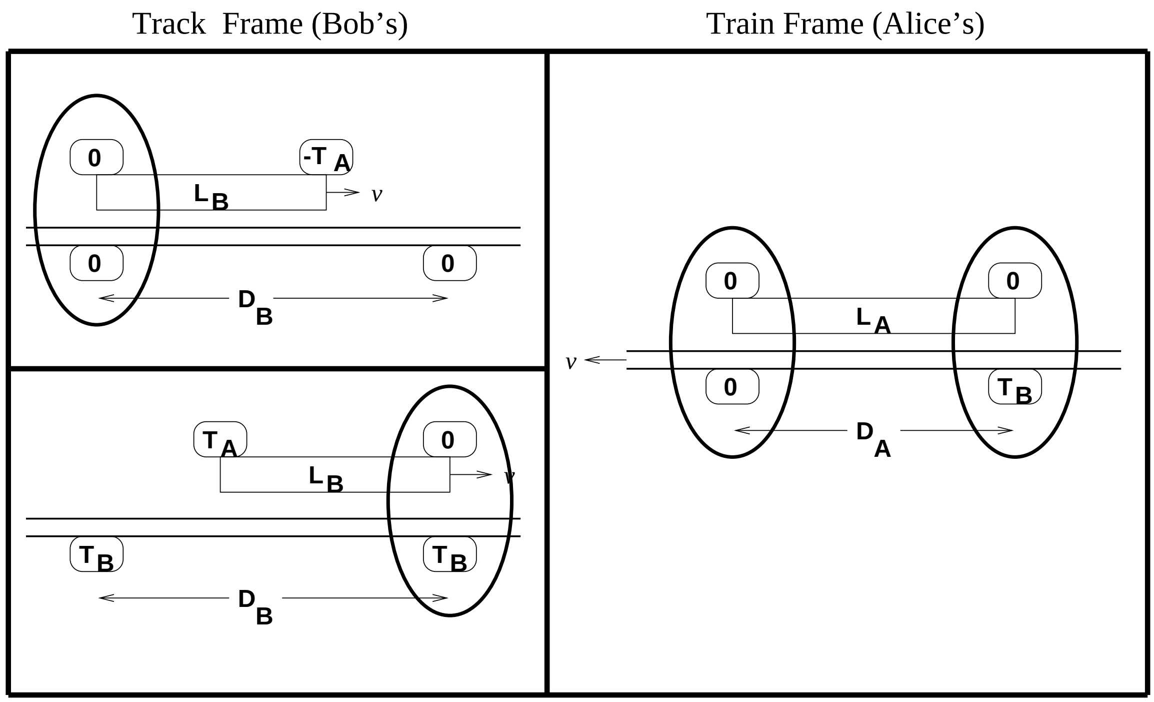

Figure 2. Figure 1 is redrawn to emphasize that although there are substantial differences of opinion between Alice and Bob (for example how long the train is, how long the stretch of track is between the two track-frame clocks, which pair of clocks is correctly synchronized, which pair of clocks is running slowly compared with which) there is complete agreement about things that happen at the same place *and* at the same time.

Thus the events that are encircled on the left of the upper track-frame picture (two clocks being together and reading 0) are described in exactly the same way in the encircled region on the left of the train-frame picture. And the two events that are encircled on the right of the lower track-frame picture (two clocks being together, the clock on the train reading 0 and the clock on the track reading T_B are also described in exactly the same way in the encircled region on the right of the train-frame picture.

There is a disagreement about whether the two events *are* (train-frame) or *are not* (track-frame) simultaneous. And there is a lot of disagreement about what is going on *somewhere else* while[18] the encircled events are happening. But there is no disagreement about the encircled events themselves.

[18] Note: "while" means "at the same time as", and is therefore a potentially treacherous word.

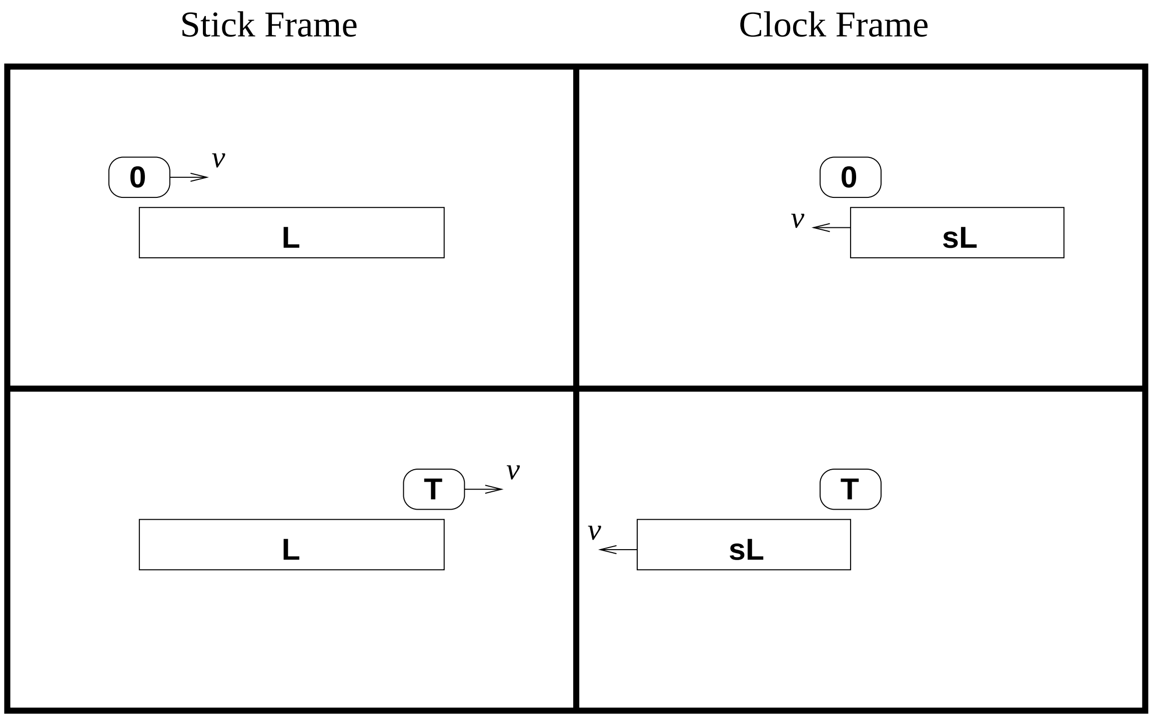

Figure 3. A stick and a clock, in relative motion. On the left we see things as described in the rest frame of the stick. The (proper) length of the stationary stick is L. The clock moves with speed v from the left end of the stick to the right end. It reads 0 when it passes the left end of the stick and T when it is passes the right end.

On the right we see things as described in the rest frame of the clock. The stick moves to the left with speed v and its length is only sL, where s is the shrinking factor. The clock is stationary. It reads 0 when the left end of the stick passes it, and T when the right end of the stick passes it. The actual value of T (in either set of pictures) is $T = sL/v$. In the clock frame this reflects the fact that the moving stick has shrunk by a factor s and the stationary clock runs at its normal ("proper") rate. In the stick frame it reflects the fact that the moving clock is running slowly by the factor s and the stationary stick has its normal ("proper") length.