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4. Relativistic Addition of Velocities

If Alice, a passenger on a train moving at v feet per second, can throw a ball at u feet per second, then if she throws the ball toward the front of the train, its speed w with respect to the tracks will be

$$w = u + v \tag{4.1}$$

in the same direction as the train.

This is known as the nonrelativistic velocity addition law. It is called "nonrelativistic" because it is only accurate when the speeds u and v are small compared to the speed of light. Evidently it fails to work when $u = c$ (i.e. if Alice turns on a flashlight instead of throwing a ball) for we know that the speed w of the light in the track frame will not be $c + v$ but simply c — the same value it has in the train frame!

Suppose, however, that Alice fired a gun that expelled "bullets" whose muzzle velocity u was 90% of the speed of light.[1] If the addition law (4.1) fails when $u = c$, it would be surprising if it worked very well when u was $0.9c$ and in fact it does not. It turns out that both (4.1) and the frame-independence of the special velocity c are special cases of a very general rule for compounding velocities that works whether or not the speeds involved are small compared to the speed of light. This "relativistic velocity addition law" states that

$$w = \frac{u + v}{1 + \left(\frac{u}{c}\right)\left(\frac{v}{c}\right)}. \tag{4.2}$$

If u and v are both small compared to the speed of light, then u/c and v/c are both small numbers. Their product is then a small fraction of a small number — i.e. a *very* small number — so the relativistic rule (4.2) differs from the more familiar nonrelativistic rule (4.1) only by dividing the nonrelativistic result by a number that hardly differs from 1. If, on the other hand, $u = c$, then (4.2) requires w also to be c, whatever the value of v may be.[2] Thus (4.2) is consistent with both our nonrelativistic experience[3] and the fact

[1] The "bullets", if you insist on getting practical about it, could be photons travelling down the train in a pipe containing a fluid in which the speed of light was only 0.9 feet per nanosecond. It is only the speed of light *in vacuum* that is the same in all frames of reference.

[2] Check this for yourself! It's an easy algebraic exercise.

[3] I.e. our experience in dealing with situations in which all relevant speeds are small compared to the speed of light.

that the speed of light is the same in all inertial frames of reference.

I shall now show you that the general relativistic rule (4.2) is a very direct consequence of the constancy of the velocity of light. We shall find that if the speed of light is the same in all inertial frames of reference then the addition law (4.1) must be replaced by (4.2) regardless of what kind of moving objects we are describing and regardless of how fast they are moving.

Before we embark on this exercise note that the explicit occurrence of the speed c in (4.2), even when none of the objects or frames of reference associated with u, v, or w have anything to do with light, gives an early indication that the speed c is built into the very nature of space and time. Objects that move at that special speed, move at that speed in all frames of reference, as a direct consequence of (4.2) itself. Photons in vacuuum happen to be examples of such objects. But the speed c has an importance that goes far beyond the fact that it is the speed at which light happens to move in empty space.

To develop a strategy for deducing the relativistic addition rule (4.2), we must first ask what goes wrong when we try to justify the nonrelativistic rule (4.1). The obvious way to determine the speed of an object is to determine the time it takes it to traverse a race-track of known length. Doing this requires two clocks, placed at the two ends of the race-track, to determine the exact times at which the object starts and finishes the race. To arrive at the nonrelativistic velocity addition law (4.1) we implicitly assumed that people using the train frame and people using the track frame would agree on whether or not those two clocks were synchronized.[4] We also assumed that the people would agree on the length of the race-track between the two clocks and on the rates at which the clocks were running. But the constancy of the velocity of light means that the nonrelativistic addition law (4.1) cannot be correct for an object moving at the speed of light, and therefore it means that some of the assumptions on which (4.1) rests must be wrong.[5]

But if we are not allowed to make such assumptions about the basic instruments with which we measure velocities, how can we deduce the correct rule for compounding velocities? One way to arrive at the correct rule would be to figure out, and then take fully into account, a set of new "relativistic" rules about clock–synchronization disagreements,

[4] Prior to Einstein the assumption was never explicitly noted. People just took it for granted that there was nothing problematic about whether two clocks in different places were synchronized.

[5] This, in turn, should make us suspicious of the validity of the nonrelativistic addition law (4.1) for any velocities at all.

rates of moving clocks, and lengths of moving measuring sticks, but this takes a bit of doing.[6]

There is a more direct way to get at (4.2), by taking advantage of the fact that we know the the speed of at least one thing — light. By being clever we can use light to help us measure the speed of anything else in a way that makes no use whatever of either clocks or measuring sticks. This will enable us to deduce the rule for how velocities change when the frame of reference changes, without assuming anything about the behavior of clocks and measuring sticks. The basic idea is simply to let the object — call it a ball — run a race with a pulse of light — call it a photon. By comparing how far the ball goes with how far the photon goes, we can figure out its speed.[7]

This neat idea runs into an immediate difficulty. Although the photon and the ball start their race in the same place they will be in different places at the end of the race. But to compare how much ground they cover during the race we must be able to determine exactly where the ball is at the precise moment the photon reaches the finish line.[8] To do this we need two synchronized clocks, one at the finish line and one with the ball. We can then determine where the ball is at the moment the photon reaches the finish line, by noting where the ball is when its clock reads exactly the same time that the clock at the finish line reads at the moment the photon gets to the finish line. But this forces us to rely on possibly unreliable clocks — precisely what we wished to avoid.

Fortunately there is an easy way to avoid this problem. We do not end the race when the photon reaches the finish line. Instead we arrange for it to hit a mirror and bounce back the way it came. We end the race only when the photon reencounters the ball, which is still moving in its original direction. By ending the race when the photon and the ball arrive at the same place, we have solved the problem of determining without clocks where the ball is along its path when the race ends. At that moment the ball is precisely where the photon is.

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[6] This, in fact, is the way in which the correct relativistic addition law (4.2) is deduced in most expositions of the subject. We will eventually construct the new set of rules about clocks and measuring sticks, but at this stage we don't know any of them. We can nevertheless figure out the correct velocity addition law before learning anything about the behavior of moving clocks and measuring sticks.

[7] If, for example the photon, moving at speed c, covers twice as much ground as the ball, then the speed of the ball must be $\frac{1}{2}c$.

[8] Let's take the case where the ball goes slower than the photon. Eventually we will see that there is something highly problematic about balls that move faster than light.

Suppose this is all done on a train. We first describe the race using the train frame. Let the race start at the rear of the train and let the photon be reflected back toward the rear when it reaches the front. Suppose the photon meets the ball a fraction f of the way from the front of the train back to the rear.[9] The photon has gone the entire length of the train *plus* an additional fraction f of that length, but the ball has only gone the entire length of the train *minus* that same fraction f of the length. The ratio of the distance covered by the ball to the distance covered by the photon is thus $\frac{1-f}{1+f}$. This must also be the ratio of their speeds.[10] So if we call the velocity of the ball in the train frame u, then since the speed of the photon in either direction is c,

$$\frac{u}{c} = \frac{1-f}{1+f}. \tag{4.3}$$

The people on the train have thus measured the speed of the ball without using clocks and without having to know the length of the cars[11] in their train!

Pause to convince yourself that (4.3) really does summarize a simple and practical way to compare the velocities of two objects, which avoids using any clocks and avoids having to know any absolute distances.

It will be useful to rewrite[12] (4.3) as a relation that expresses the fraction f in terms of the speed u of the ball and the speed of light c:

$$f = \frac{c-u}{c+u}. \tag{4.4}$$

[9] For example if the train consists of 100 identical cars (numbered 1,2,3,... starting from the front) and the photon meets the ball in the passageway between cars 34 and 35, then $f = 0.34$.

[10] For example if the ball covers 1/5 the distance the photon covers, then its speed must be 1/5 the speed of the photon.

[11] They only have to be able to count cars. If the ball met the photon some fraction of the way along a car, they would have to be able to compare the lengths of the two parts of the car, but they could do this without knowing the absolute length of either part by just counting up the number of times some measuring stick (of unknown length) went into both parts.

[12] Whenever I make an assertions that two expressions are equivalent (in this case the relation (3) and the relation (4)) you should always do the algebra (on a piece of paper or in your head) to convince yourself that I got it right. If the algebraic challenge proves too great, at least convince yourself that (4.3) and (4) are consistent by checking a few special cases. For example if $f = \frac{1}{2}$ then (4.3) tells us that $u/c = \frac{1}{3}$. On the other hand when $u = \frac{1}{3}c$, (4) does indeed give $f = \frac{1}{2}$.

Now let us start all over again and analyze a similar race on the train, but this time using the frame of reference of the track, in which the train has a velocity[13] v and the ball, a velocity w. As before, the photon and ball both start at the rear of the train, the photon reaches the front first, bounces back toward the rear, and the race ends when the photon reencounters the ball. We again want to know what fraction of the way back along the train the photon has to go before it meets the ball. We want to express this fraction entirely in terms of the various speeds. This time the analysis is a bit more complicated, since the train is moving while the race goes on.

We continue to assume that the photon moves with speed c in both directions in the track frame. In a little while we are going to appeal to the constancy of the velocity of light to interpret this as exactly the same race as the one we analyzed in the train frame. Meanwhile, however, it might be a good idea to put the first race out of your mind while analyzing this one. You may think, if you want, of the photon in the second race as a new "track-frame photon" which has the speed c in the track frame, unlike the old train-frame photon, which had the speed c in the train frame. If you look at it this way (and you should for now) then there is nothing at all peculiar about the track-frame analysis that follows. It's more elaborate than the train-frame analysis, but only because now the train is moving too, which complicates things.

To analyze the race in the track frame we shall have to talk about track-frame distances and times. We shall not, however, make any assumptions about how track-frame clocks and measuring sticks behave except that track-frame people have taken all necessary precautions to ensure that the track-frame speed of an object is indeed the track-frame distance it goes in a given track-frame time. Our goal is to end up with relations like (4.3) or (4.4) that involve no times and lengths. The relation we seek involves only velocities, along with the fraction f of the way back along the train the photon has to go before it meets the ball.[14]

Suppose it takes a time T_0 for the photon to get from the back of the train to the mirror at the front and a time T_1 for the reflected photon to get from the front to the point a fraction f of the way back along the train where it reencounters the ball. Let L be the length of the train and let D be the distance between the front of the train and the ball

[13] We take u, v, and w all to be positive — i.e. the ball moves to the right in the train frame, and the train and ball move to the right in the track frame — so that velocities and speeds are the same; the result we arrive at, however, turns out to be valid for positive or negative velocities.

[14] See part (3) of Figure 1 on the last page.

at the moment the photon reaches the front of the train.[15,16]

Since T_0 is the time it takes the photon to get a distance D ahead of the ball and since both start in the same place, moving toward the front with speeds c and w, we must have

$$D = cT_0 - wT_0. \tag{4.5}$$

On the other hand T_1 is the time it takes the photon and ball, initially a distance D apart, to get back together. Since the photon covers a distance cT_1 during this time and the ball, wT_1, we have

$$D = cT_1 + wT_1. \tag{4.6}$$

Since we don't know the value of D we shall eliminate it from these two relations. This gives us $cT_0 - wT_0 = cT_1 + wT_1$, which it is convenient to write in the form

$$\frac{T_1}{T_0} = \frac{c - w}{c + w}. \tag{4.7}$$

But of course we don't know the times T_1 and T_0 either. There is, however, a second quite similar way to get at the same ratio of these two times, by comparing what the photon does, not to what the ball does, but to what the train does. Note first that T_0 is the time it takes the photon to get ahead of the rear of the train by the track-frame length of the train L. Since the photon has speed c and the train, speed v,

$$L = cT_0 - vT_0. \tag{4.8}$$

Note next that T_1 is the time it takes the photon, moving toward the rear at speed c to meet a point on the train originally a distance fL away from it that moves toward it at velocity v. Thus

$$fL = cT_1 + vT_1 \qquad (4.9).$$

We don't know the actual value of L any more than we knew the actual value of D, but we can also eliminate L from these last two equations. This gives us $cT_1 + vT_1 =$

[15] Of course these times and distances are all unknown track-frame times and distances. But since the reasoning that follows is entirely track-frame reasoning, and since the problematic quantities D, L, T_0, and T_1 all drop out of the final result, this causes us no difficulty.

[16] It is extremely important for you now to inspect the figure on the last page, and to keep referring to it in the course of the assertions that follow. You should also read the caption of that figure, checking what it says against the figure itself. Only after that should you start to read the argument that follows below.

$f(cT_0 - vT_0)$, which gives us a second expression for the ratio of T_1 to T_0:

$$\frac{T_1}{T_0} = f\Big(\frac{c-v}{c+v}\Big). \tag{4.10}$$

Although we don't know either T_1 or T_0 this expression for their ratios must be the same as the other expression (4.7). We conclude that

$$f\Big(\frac{c-v}{c+v}\Big) = \frac{c-w}{c+w}. \tag{4.11}$$

This is the relation we need. All unknown times and distances have dropped out and we have a relation involving only the fraction f and some velocities. It follows immediately from (4.11) that the fraction f can be expressed in terms of the velocities v and w by

$$f = \Big(\frac{c+v}{c-v}\Big)\Big(\frac{c-w}{c+w}\Big). \tag{4.12}$$

I stress that as a piece of track-frame analysis, applicable to a race between a ball with track-frame speed w and a photon with track-frame speed c, both on a train with track-frame speed v, there is nothing at all peculiar about the analysis leading to (4.12).[17] Galileo would have been quite happy with it.[18] Indeed, the result (4.12) remains entirely correct if we replace the photon by anything at all that moves with the same speed in both directions, and allow that speed, c, to be any speed at all greater than w and v.

We do something that would not have been to Galileo's liking only when we now declare that if the photon really *is* a photon, and if the speed c really is the speed of light in vacuum, then these two pieces of analysis we have now completed, can be taken to be train-frame and track-frame analyses of *one and the same race.* In this race u is the train-frame velocity of the ball, w is the track-frame velocity of that same ball, and v is the track-frame velocity of the train. Peculiarly, however, — and this is the *only* peculiarity in the entire argument — we now insist that the track-frame speed c of that one photon (in either direction) is exactly the same as the train-frame speed c of that same photon

[17] As a reassuring check that we haven't made some mistake in getting to (4.12), notice the following: Suppose the velocity v of the train in the track-frame were 0. Then the track frame would be the same frame as the train frame. Consequently w, the velocity of the ball in the track frame, would be the same as u, the velocity of the ball in the train frame. And indeed, if you set v to zero and take w to be u, you do get back our old train-frame result (4.4).

[18] Provided we made the train a boat.

(in either direction). In both frames (and in both directions) that speed is one foot per nanosecond. This is the only place in the argument where we invoke the counterintuitive principle of the constancy of the velocity of light.

But if we have indeed been describing one and the same race in two different frames then f, the fraction of the way back from the front of the train where the photon meets the ball, must have the same value in either frame. For although there might (and indeed, as we shall see, there will) be disagreement between the two frames of reference over the length of the cars of the train, there can be no disagreement about where on the train the photon meets the ball. Their reunion could trigger an explosion, for example, that would make a smudge on the floor, which all observers in all frames could inspect later on at their leisure to confirm in which part of which car the meeting took place.

So the track-frame expression (4.12) for the fraction f must agree with the train-frame expression (4.4). Setting them equal gives us a relation between the three velocities w, u, and v:

$$\left(\frac{c+v}{c-v}\right)\left(\frac{c-w}{c+w}\right) = \frac{c-u}{c+u}. \tag{4.13}$$

It is useful to rewrite this relativistic velocity addition law in a form (like the form of (4.1), the nonrelativistic velocity addition law) in which w appears on the left side and u and v on the right:

$$\frac{c-w}{c+w} = \left(\frac{c-u}{c+u}\right)\left(\frac{c-v}{c+v}\right). \tag{4.14}$$

This is the relativistic rule that replaces the nonrelativistic rule (4.1). Instead of *adding* u and v to get w we must *multiply* an expression involving u by an expression of the same form involving v to get a third expression of the same form involving w.

The relation between the nonrelativistic rule (4.1) and the relativistic rule (4.14) is not at all clear. To see that they are, in fact, rather simply related, one must carry out the simple algebraic exercise[19] of solving (4.14) for the velocity w of the ball in the track frame in terms of its speed u in the train frame and the speed v of the train. The result is the relativistic "addition law" stated in (4.2) above:

$$w = \frac{u+v}{1+\left(\frac{u}{c}\right)\left(\frac{v}{c}\right)}. \tag{4.15}$$

Although the two forms (4.14) and (4.15) of the velocity addition law are just different ways of expressing the same relation among the three velocities w, u, and v, it is helpful

[19] Which is done for you in the Appendix at the end of this essay, in case you find it too complicated to do for yourself.

to keep them both in mind, since one form can be more useful than the other, depending on the question one is asking. Thus the form (4.15) makes immediately evident (as noted on page 2 above) why the nonrelativistic addition law $w = u + v$ becomes quite accurate when u and v are small compared to the speed of light. The form (4.14), on the other hand, reveals the following important fact:

If the speed u of the ball in the train frame and the speed v of the train in the track frame are both less than the speed of light, then both $\frac{c-u}{c+u}$ and $\frac{c-v}{c+v}$ will be numbers between 0 and 1. Since the product of two numbers between 0 and 1 is also between 0 and 1, this means that $\frac{c-w}{c+w}$ is between 0 and 1, which implies in turn that the speed w of the ball in the track frame is also less than the speed of light.[20] Thus the obvious stratagem for producing an object moving faster than light does not work: if you have a cannon that shoots balls at 90% of the speed of light, and you put it on a train moving at 90% of the speed of light, then the speed of the ball in the track frame will still be less than the speed of light. Indeed in this particular case (4.15) tells us that the speed w of the ball in the track frame will be a fraction $\frac{0.9+0.9}{1+(0.9)^2} = \frac{1.80}{1.81}$ of the speed of light — about 99.45%. This is the first indication we have found — there will be others — that no material object can travel faster than the speed of light.

For many purposes it is helpful to abstract the relativistic addition law from the context of balls, trains, and tracks, and state it in terms of the velocities of certain objects (or frames of reference) with respect to other objects (or frames of reference). Let us regard the track as an object called A, the train as an object called B, and the ball as an object called C. The velocity v of the train in the track frame we now call v_{BA} — "the velocity of B with respect to A". In the same way we call the velocity u of the ball in the train frame v_{CB}, the velocity of C with respect to B, and we call the velocity w of the ball in the track frame, v_{CA}. In this language the two forms for the addition law become

$$\frac{c - v_{CA}}{c + v_{CA}} = \Big(\frac{c - v_{CB}}{c + v_{CB}}\Big)\Big(\frac{c - v_{BA}}{c + v_{BA}}\Big). \tag{4.16}$$

and

$$v_{CA} = \frac{v_{CB} + v_{BA}}{1 + \big(\frac{v_{CB}}{c}\big)\big(\frac{v_{BA}}{c}\big)}. \tag{4.17}$$

Another advantage of (4.16) over (4.17) emerges when you consider the case in which object C is a rocket that itself emits a fourth object D. If D has speed v_{DC} with respect

[20] The conclusion that if u and v are less than c then so is w is not as immediately evident from (4.15) as it is from (4.14).

to C what is the speed v_{DA} of D with respect to A? In other words, what form does the addition law take when we compound three speeds instead of just two? This leads to a great mess if we try to answer the question using the form (4.17), but if we use the addition law in the form (4.16) we merely note the following:

The speed of D with respect to A can be arrived at by compounding the speed of D with respect to C and the speed of C with respect to A. Applying the general rule (4.14) to this case gives

$$\frac{c - v_{DA}}{c + v_{DA}} = \left(\frac{c - v_{DC}}{c + v_{DC}}\right)\left(\frac{c - v_{CA}}{c + v_{CA}}\right). \tag{4.18}$$

But now we can apply (4.14) again to express the quantity containing v_{CA} in terms of v_{CB} and v_{BA} to get

$$\frac{c - v_{DA}}{c + v_{DA}} = \left(\frac{c - v_{DC}}{c + v_{DC}}\right)\left(\frac{c - v_{CB}}{c + v_{CB}}\right)\left(\frac{c - v_{BA}}{c + v_{BA}}\right). \tag{4.19}$$

So to compound three speeds rather than just two, we just put a third term into the product in (4.16) to get (4.19). Evidently if D were a rocket that emitted a fifth object E, we could continue in this way, and so on indefinitely. The rule in the form (4.17) would get more and more complicated, but in the form (4.16) it would retain the same simple form.

The addition law in either of its two forms (4.17) or (4.16) continues to hold even when not all the velocities have the same sign (e.g. even when the ball moves toward the rear of the train, rather than the front). If, for example, Alice throws a ball with speed u toward the *rear* of a train that moves with positive velocity v along the track, then the velocity w of the ball along the track is given by

$$w = \frac{-u + v}{1 - \frac{u}{c}\frac{v}{c}} \tag{4.20}$$

since this is what (4.2) reduces to when u is replaced by $-u$. It is a useful exercise to check this by repeating the analysis of this essay for the case where the race starts at the front of the train rather than at the rear.

An easier, if more abstract way to see that (4.16) and (4.17) remain valid even when all the velocities are not positive, is to note that although we have derived (4.16) in the case where all three of the velocities v_{CA}, v_{CB} and v_{BA} are positive, we can introduce negative velocities by exploiting the general fact that

$$v_{YX} = -v_{XY} \qquad (4.21).$$

We can, for example, replace the positive v_{BA} in (4.16) by the negative velocity $v_{AB} = -v_{BA}$, and then algebraically transform (4.16) into:

$$\frac{c - v_{CB}}{c + v_{CB}} = \Big(\frac{c - v_{CA}}{c + v_{CA}}\Big)\Big(\frac{c - v_{AB}}{c + v_{AB}}\Big). \tag{4.22}$$

Notice that this has exactly the same form as (4.16) — only the labels A and B have been interchanged. But now one of the three velocities (v_{AB}) is negative.

Similar tricks using (4.21) enable one to reexpress (4.16) in other equivalent forms in which either or both of the two velocities on the left are negative.

Appendix

Write (4.14) in the form[21]

$$\frac{c - w}{c + w} = \frac{a}{b}, \tag{4.23}$$

where

$$a = (c - u)(c - v) \tag{4.24}$$

and

$$b = (c + u)(c + v). \tag{4.25}$$

It follows from (4.23) that

$$(c - w)b = (c + w)a \tag{4.26}$$

or

$$c(b - a) = w(b + a) \tag{4.27}$$

so that

$$\frac{w}{c} = \frac{b - a}{b + a}. \tag{4.28}$$

Now according to (4.25) and (4.26)

$$\begin{aligned} b =& c^2 + c(u + v) + uv. \\ a =& c^2 - c(u + v) + uv \end{aligned} \tag{4.29}$$

and therefore

$$\begin{aligned} b + a =& 2(c^2 + uv) = 2c^2\big(1 + (u/c)(v/c)\big), \\ b - a =& 2c(u + v) \end{aligned} . \tag{4.30}$$

These two relations in (4.30) immediately reduce (4.28) to (4.15).

[21] The reason for doing this is simply that a and b are easier to carry through the next few steps than the more complicated expressions that they stand for.