

11. Mass, Momentum, and Energy

It is necessary to conclude a series of lectures on relativity with a few on $E = Mc^2$, possibly the most famous equation of all time.^{1,2} To understand these we will have to examine a third quantity, momentum (P).³

NONRELATIVISTIC MASS, MOMENTUM, AND ENERGY

What is mass?

Let's begin by examining how mass is defined nonrelativistically.⁴ This turns out to be very useful since the nonrelativistic definition survives in the relativistic case too, provided one adds a small footnote.⁵

Here are two ways *not* to define mass.

1. *First bad definition of mass (Newton)*. Newton defined mass to be “quantity of matter.” This is useless for two reasons:

(a) How do you count up the quantity of matter in something? If matter was all built out of identical little bricks one might be able to do it by counting the number of bricks, but unfortunately matter (as we understand it today) is made up of many different kinds of bricks, so this doesn't work unless you have an independent definition for the “quantity of matter” in two bricks of different types.

(b) Furthermore even when you put together identical bricks it turns out that the mass of the object you construct depends on how you put the bricks together.

¹ Except, perhaps, for $C^2 = A^2 + B^2$, which we have also been able to put to good use.

² Because these notes are somewhat heavy with equations, I have tried to summarize the argument with a bare minimum of equations in Appendix C at the end. The proper strategy is to read the main notes first, and then read Appendix C to check that you understood everything. But you might also find it helpful to glance at Appendix C at the start to get an overall picture of where we are heading.

³ I have no idea why momentum is always denoted by P or p , except that it has to be something other than M or m , which are reserved for mass.

⁴ “Non-relativistically” does not mean “ignoring the principle of relativity”; on the contrary, it means maintaining the principle of relativity, but examining how it operates when the speeds are all small compared with the speed of light c . We shall turn to the relativistic case, in which speeds are not restricted to be small compared with that of light, on p. 8 below.

⁵ Which we shall do at the appropriate moment.

2. *Second bad definition: Mass is weight.* The badness of this definition forms a tedious part of all introductory physics courses. The problem is that the weight of an object is the force that gravity exerts on it. This depends on where the object is. An object's weight on the moon is about a sixth of its weight on earth (which itself varies a bit depending on where you are on the earth) and its weight in empty space is zero.

3. *Correct definition.* The mass of an object is a measure of how hard it resists attempts to change its velocity. Under a given set of circumstances, the bigger the mass of an object, the less its velocity changes.

This is far too informal a statement to stand by itself, but it captures the essential quality of the concept of mass.⁶ It's easy to push around a beachball, harder to push around a solid wooden ball of the same size, and extremely hard to push around a solid lead ball of that size. To make the concept of mass less qualitative, we must go beyond this informal definition and state some simple facts, which make it possible to give a quite precise definition of mass. To do this we must return to the kinds of collisions we were considering at the very beginning, in which two particles come together with certain velocities,⁷ collide, and go off with certain other velocities. The correct and precise definition of mass is contained in the following crucial fact about such collisions:

It is possible to associate with every particle a positive number m , called its mass, which is a measure of how little the velocity of the particle changes in such collisions; the bigger the mass, the less the change in velocity. To give the precise relation between the masses of the two particles and their changes in velocity, call the particles 1 and 2, and their masses m_1 and m_2 . Call their velocities before the collision \mathbf{u}_1^b and \mathbf{u}_2^b , and their velocities after the collision \mathbf{u}_1^a and \mathbf{u}_2^a , so that the *changes* in their velocities are $\mathbf{u}_1^a - \mathbf{u}_1^b$ and $\mathbf{u}_2^a - \mathbf{u}_2^b$. Many experiments establish the useful fact that the comparative size of the two changes in velocity is entirely determined by the comparative size of the two masses, according to the simple rule:

$$\frac{\mathbf{u}_1^a - \mathbf{u}_1^b}{\mathbf{u}_2^a - \mathbf{u}_2^b} = -\frac{m_2}{m_1}. \quad (1)$$

Several comments:

⁶ Note that by defining mass in terms of velocities, so that measuring a mass reduces to measuring times and distances in an appropriate experiment, the correct definition is ideally suited for a reexamination in the relativistic case.

⁷ We must now take care to distinguish explicitly between velocity and speed. I shall follow the widespread practice of using bold face letters (\mathbf{u}) for velocities, which can be positive or negative, and italic letters (u) for speeds, which are always positive. Thus a particle with speed u has velocity $\mathbf{u} = u$ if it moves down the tracks, and $\mathbf{u} = -u$ if it moves up the tracks. Note that the square of a velocity is the same as the square of the corresponding speed: $u^2 = \mathbf{u}^2$.

1. Since the masses are positive, the minus sign simply means that the ratio of the changes in velocities is negative—*i.e.* the change in the velocity of one of the particles is in the opposite direction from the change in velocity of the other. If the velocity of one increases, the velocity of the other decreases.⁸

2. In the nonrelativistic theory this rule holds whatever the individual velocities happen to be. One might expect there to be trouble when the speeds approach relativistic values (significant fractions of a foot per nanosecond), and indeed the rule then fails to hold, as we shall see. However even in the more accurate relativistic theory, as one also might expect — indeed, as one ought to require — the rule holds to a very high degree of precision provided all particle speeds are small compared with the speed of light. This makes it possible to use the nonrelativistic definition of mass emerging from (1) to define mass even in the relativistic theory of energy and momentum. One simply makes one additional proviso: all the particle speeds in a collision designed to compare the masses of two particles, must be small compared with the speed of light.⁹

3. Implicit in the definition (1) of mass is the fact that the same number m works for a given particle regardless of what other particle it collides with. Thus although our definition gives only the comparative resistance to changes of velocity of a pair of particles, we end up with the same collection of masses for all the particle regardless of which pairs we choose to test against each other.^{10,11}

4. Note that this definition of mass is consistent with the principle of relativity. The numbers you get for the mass ratios do not depend on the frame of reference in which

⁸ Note that increasing or decreasing *velocity* is not the same as increasing or decreasing *speed*. If a particle moving to the right slows down a little its velocity decreases. But if a particle moving to the left speeds up its velocity also decreases, because it becomes a larger negative number. And if a particle moving to the left slows down a little its velocity increases, because it becomes a smaller negative number.

⁹ “How small?” you might ask. That depends on how accurately you want to know the ratio of the masses. Since no mass is known to better than about ten significant figures, an error of one part in ten billion is good enough for all practical purposes which, as we shall see, means the speeds ought to be less than a hundred thousandth of the speed of light, or about 10 feet per millisecond — roughly 10 times the speed of sound in air — still a pretty brisk clip.

¹⁰ Thus by testing 1 and 2 we learn the ratio m_2/m_1 and by testing 2 and 3 we learn m_3/m_2 . The product of these two ratios is m_3/m_1 and indeed, if we test 1 and 3 directly this is precisely what we get. There is nothing in the nature of collision experiments that logically requires that this should be so. It is an important fact about nature that different kinds of particles behave in this very simple way when they collide at nonrelativistic speeds.

¹¹ Of course we can only determine in this way the ratio of the masses of all the particles. The overall scale is arbitrary, and can be fixed, for example, by taking one standard object and declaring its mass to be “one kilogram”.

the collision is described, *provided* we use the *nonrelativistic* velocity addition law. For if we view all the collisions in a frame moving to the right with speed v —*i.e.* with a velocity¹² \mathbf{v} that is positive—then every velocity \mathbf{u} appearing in (1) is replaced by $\mathbf{u} - \mathbf{v}$, which leaves *changes* in velocity, which are all that appear in (1), unaffected.¹³ That the definition works in any inertial frame of reference is, of course, crucial if it is to be viewed as embodying a law of nature, for the principle of relativity requires laws of nature to hold in all inertial frames.

What is momentum?

With a little elementary algebraic manipulation we can rewrite (1) in the mathematically equivalent form:

$$m_1 \mathbf{u}_1^b + m_2 \mathbf{u}_2^b = m_1 \mathbf{u}_1^a + m_2 \mathbf{u}_2^a. \quad (2)$$

Although this has precisely the same mathematical content as (1), it presents the information in a somewhat different way. The left side of (2) only contains velocities before the collision, while the right side only contains velocities after. We have therefore discovered a quantity that is unchanged, or “conserved”, by the collision. It is called the total momentum, usually denoted by the symbol \mathbf{P} . We call it “total” momentum because it is convenient also to define the momentum \mathbf{p} of an individual particle of mass m and velocity u by

$$\mathbf{p} = m\mathbf{u}, \quad (3)$$

so that the total momentum \mathbf{P} of two particles is just

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2. \quad (4)$$

Eq. (2) is called the “Law of Conservation of Momentum”. From our point of view it is just a reformulation of our definition of mass. But like that “definition” it has profound physical content going well beyond a merely conventional definition. It is a remarkable *fact*

¹² Do not confuse the velocity \mathbf{v} of the new frame of reference with the velocities \mathbf{u} of the particles participating in the collision: \mathbf{v} is fixed throughout the collision and has nothing to do with the collision itself. It is merely the relative velocity of the two frames whose descriptions of the collision we are interested in comparing. The individual particle velocities \mathbf{u} , on the other hand, can vary from one particle to another and can change in the course of the collision.

¹³ This provides another clue that something goes awry in the relativistic case, for the relativistic rule is that when you change frames of reference \mathbf{u} is replaced by $\frac{\mathbf{u}-\mathbf{v}}{1-\mathbf{u}\mathbf{v}/c^2}$. Of course if both speeds u and v are small compared with c this difference is so small as to be unimportant.

that it is *possible* to assign to every particle a number m in such a way that momentum is indeed conserved in all collisions between all possible pairs of particles.

Conservation of momentum continues to hold under conditions even more general than those I have just described. Not surprisingly, it continues to hold when more than two particles participate in the collision. It also continues to hold even when the motion of the particles is not confined to a single line.¹⁴ Somewhat more surprisingly, it continues to hold even when the numbers or kinds of particles *change* as a result of their collision. Suppose, for example, particles 1 and 2 stick together to form a single new particle, particle 3. When this happens the mass of particle 3 turns out to be just the sum of the masses of the original two, and momentum continues to be conserved.

Note, in this case, that it is crucial that m_3 should be $m_1 + m_2$. Otherwise momentum could not be conserved in all frames of reference. For if all the velocities \mathbf{u} are replaced by $\mathbf{u} - \mathbf{v}$, then the momentum before the collision is reduced by $(m_1 + m_2)\mathbf{v}$, while the momentum after the collision is reduced by $m_3\mathbf{v}$. Thus if m_3 were not $m_1 + m_2$, momentum would not be conserved in the new frame. This is so important that it is stated as a Law of Conservation of Mass:¹⁵ if two particles m_1 and m_2 merge into a single particle of mass M , then

$$M = m_1 + m_2. \tag{5}$$

If the Law of Conservation of Mass did not hold, then the Law of Conservation of Momentum could not hold either.

What is energy?

We started off interested not only in mass M , but also in energy, E . To see how E enters the picture, it is very useful to examine a two-particle collision in a very special frame of reference, in which the total momentum is zero. In this zero-momentum frame¹⁶ we have before the collision

$$m_1\mathbf{u}_1^b + m_2\mathbf{u}_2^b = 0 \tag{6}$$

and, because momentum is conserved,

$$m_1\mathbf{u}_1^a + m_2\mathbf{u}_2^a = 0 \tag{7}$$

¹⁴ In that case one must specify the velocity of a particle by its components along three different directions (for example, up–down velocity, north–south velocity, and east–west velocity). The generalized law then says that momentum is independently conserved for each of these three different components.

¹⁵ Our goal, $E = mc^2$, is related to the fact that this law too must often fail in the relativistic case, as we shall see.

¹⁶ A term preferred by physicists is “center of mass frame”, but we shall use the more directly descriptive name.

after the collision too. In the zero-momentum frame the particles move in opposite directions, since the velocities of 1 and 2 have to have opposite signs if their momenta add up to give zero. So in the zero-momentum frame the particles come together and then fly apart with speeds whose ratios are the same both before, and after the collision:

$$\frac{u_2^b}{u_1^b} = \frac{m_1}{m_2} = \frac{u_2^a}{u_1^a}. \quad (8)$$

But although the *ratios* of the speeds are the same both before and after the collision, there is nothing in the law of conservation of momentum to require the speeds *individually* to stay the same. Momentum conservation is perfectly consistent with both speeds either increasing or decreasing, as long as the percentage increase or decrease is the same for both particles. There is, however, something special about a collision in which the speeds themselves remain the same—*i.e.* in which the particles simply bounce back in the directions they came from with their original speeds. One calls such collisions “elastic,” and calls “inelastic” those collisions in which the individual speeds change. An inelastic collision in which both speeds dropped might be one in which the particles tended to stick together when in contact, and therefore lost some of their speed in the course of pulling apart again. An inelastic collision in which both speeds increased might be one in which a small explosive charge was set off when the particles touched, propelling them back faster than they came together.¹⁷

Whatever the reason for a collision being elastic or inelastic, however, one singles out elastic collisions for special treatment, because in an elastic collision something else, besides momentum, is conserved. In the zero-momentum frame of two particles, it is the individual speeds themselves that are conserved, but that is special to both the zero-momentum frame and the case of two particles. It is, however, easy to see what the new quantity must be if we want it to be conserved in *all* frames of reference. Define the “kinetic energy” k of a particle of mass m and velocity \mathbf{u} by¹⁸

$$k = \frac{1}{2}m\mathbf{u}^2, \quad (9)$$

and define the total kinetic energy of two particles to be

$$K = k_1 + k_2. \quad (10)$$

¹⁷ It is an important fact that momentum continues to be conserved even in cases like these.

¹⁸ The factor $\frac{1}{2}$ is entirely a matter of convention, designed to make things come out simpler further on. Clearly we could redefine any of these quantities (m , p , or k) by introducing arbitrary numerical scale factors that were the same for all particles.

Since u_1 and u_2 are *separately* conserved in an elastic collision in the zero-momentum frame, so are k_1 and k_2 and hence their sum. Any number of other possible definitions of K would share this simple property. What makes the particular definition (9) special is that if K is conserved in *one* frame of reference it will necessarily be conserved in *all* frames. If we can establish this then we do not need to use the zero-momentum frame to check on whether or not a collision is elastic. We only need to compute $K = \frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2$ both before and after the collision; the collision is elastic if and only if K is the same before and after.

So how does K change when we change frames? The velocity \mathbf{u} changes to $\mathbf{u} - \mathbf{v}$, so the kinetic energy $k = \frac{1}{2}m\mathbf{u}^2$ changes to

$$k' = \frac{1}{2}m(\mathbf{u} - \mathbf{v})^2 = \frac{1}{2}m\mathbf{u}^2 - m\mathbf{u}\mathbf{v} + \frac{1}{2}m\mathbf{v}^2 = k - \mathbf{p}\mathbf{v} + \frac{1}{2}m\mathbf{v}^2. \quad (11)$$

If we have two particles, we just add up the changes in kinetic energy for each of them, so the total kinetic energy in the new frame is

$$K' = K - \mathbf{P}\mathbf{v} + \frac{1}{2}M\mathbf{v}^2, \quad (12)$$

where \mathbf{P} is the total momentum and M is the total mass. Suppose the kinetic energy in the original frame K is the same before and after the collision. Then since the total momentum \mathbf{P} and the total mass M are also conserved, it follows that the kinetic energy K' in the new frame will also be the same before and after the collision. Thus it is a consequence of the conservation of total momentum and total mass, that if total kinetic energy is conserved in one frame, it will be conserved in all frames. If we define a collision to be elastic if kinetic energy is conserved in the collision, then whether or not a collision is elastic is independent of frame of reference.

Summary of the nonrelativistic conservation laws

Here is a summary of the nonrelativistic state of affairs described above:

Mass. We associate with each particle a mass m which is a number characteristic of the particle, independent of the frame of reference in which the particle is described; the total mass M of a collection of particles is just the sum of their individual masses. Total mass is conserved in all collisions. Total mass is also the same in all frames of reference. Putting this last remark more formally, if M is the mass in one frame and M' is the total mass in a frame moving with velocity \mathbf{v} , then

$$M' = M. \quad (13)$$

Momentum. If a particle of mass m has a velocity \mathbf{u} we define its momentum \mathbf{p} by

$$\mathbf{p} = m\mathbf{u}. \quad (14)$$

The total momentum \mathbf{P} of a collection of particles is just the sum of their individual momenta. The total momentum is conserved in all collisions. The momentum \mathbf{P}' in a frame moving with velocity \mathbf{v} is related to the momentum \mathbf{P} in the original frame by

$$\mathbf{P}' = \mathbf{P} - M\mathbf{v}. \quad (15)$$

where M is the total mass.

Energy. If a particle of mass m has a velocity \mathbf{u} we define its kinetic energy k by

$$k = \frac{1}{2}m\mathbf{u}^2. \quad (16)$$

The total kinetic energy K of a collection of particles is just the sum of their individual kinetic energies. The total kinetic energy is only conserved in a special kind of collision, known as an elastic collision. The kinetic energy K' in a frame moving with velocity \mathbf{v} is related to the kinetic energy K in the original frame by

$$K' = K - \mathbf{P}\mathbf{v} + \frac{1}{2}M\mathbf{v}^2. \quad (17)$$

where M is the total mass and \mathbf{P} is the total momentum in the original frame.

Note here the interplay between conservation laws (quantities which are the same before and after the collision) and transformation rules (which tell how quantities change from one frame of reference to another). A conservation law relates the value of a quantity before the collision to its value after the collision, when both values are computed in the same frame of reference. For it to be a *law* it must be valid in all frames of reference, so we must use the transformation laws to check that a candidate for a conservation law is capable of being obeyed in all frames of reference. In the case of mass conservation that is easy, since mass is the same in all frames of reference. Momentum can be conserved in all frames of reference because it obeys the transformation rule (15) *and* because the total mass is the same before and after a collision. Kinetic energy can be conserved in all frames of reference (if it is conserved in any one frame) because it obeys the transformation rule (17) *and* because *both* the total momentum *and* the total mass are the same before and after a collision.

Note also the important fact that the *contingently* conserved quantity, K , does not appear in the transformation rules governing the quantities \mathbf{P} and M that are *always* conserved. If K did appear in the transformation rules for either \mathbf{P} (or M), then since K is not always conserved, neither could \mathbf{P} (or M) always be conserved.

RELATIVISTIC MASS, MOMENTUM, AND ENERGY

When we get to speeds comparable to the speed of light, this simple nonrelativistic picture falls apart. As already noted, the pleasing compatibility of these conservation

laws and their ability to be satisfied in all frames of reference makes critical use of the nonrelativistic velocity addition law, $\mathbf{u}' = \mathbf{u} - \mathbf{v}$. When this rule is seriously violated, then conservation of momentum ceases to be a rule that holds in all frames if it holds in any one, because the simple transformation rule (15) for momentum is no longer valid. The same problem arises with kinetic energy. This is not surprising. There is no reason to expect that the appropriate forms for the momentum and kinetic energy of a particle should be identical to the forms they have in the nonrelativistic case. After all, not even the rate of a moving clock or the length of a moving stick is the same as it is in the nonrelativistic case. The question we must address is whether it is possible to find new conservation laws involving suitable generalizations of the nonrelativistic definitions of mass, momentum, and kinetic energy.

These generalizations must have two crucial features: (a) They must reduce to the nonrelativistic forms when the speeds of the particles are small compared with the speed of light, since we know the nonrelativistic conservation laws hold to a high degree of accuracy in that limit; (b) If the appropriately generalized quantities are conserved in one frame of reference then they must be conserved in all frames of reference, or we could distinguish between different inertial frames of reference by doing an experiment to see whether, for example, momentum was or was not conserved.

The proper relativistic definition of mass is the easiest to deal with. As remarked upon above, we retain exactly the same definition of mass as in the nonrelativistic theory, only adding the proviso that the velocities of all particles in a collision used to determine their masses should be small compared with the velocity of light.^{19,20} As so defined, the mass of a particle continues to be an inherent property of the particle, having nothing to do with how fast the particle might be moving in collisions it might subsequently find itself in. It is an invariant, independent of frame of reference.²¹

We defer for the moment the question of whether or not total mass, defined as the sum of the masses of the individual particles, continues to be conserved in collisions that change the numbers and types of particles. Note, though, that any failure of mass conservation

¹⁹ How small, as noted, depends on how accurately we want to determine the masses. A good practical criterion is to say that they should be so small that if we repeat the experiment with even smaller velocities, we get exactly the same set of masses to within the accuracy of the method we use to determine the relevant speeds.

²⁰ But. you may protest, that does us no good in determining the mass of a photon, since photons in empty space cannot move at any speed other than the speed of light. I return to the special case of photons at the end of this essay.

²¹ If there were a particle whose mass was not invariant, then we could distinguish one inertial frame from another by performing in each frame a low velocity collision that determined the mass of the particle.

had better be by a very small amount when the speeds of all particles participating in the collision are small compared with the speed of light, since the nonrelativistic theory, in which total mass *is* conserved, holds to a high degree of precision when all speeds are small compared with c .

We turn next to the relativistic definition of the momentum of a particle of mass m . Since m continues to be simply an invariant number, characterizing the particle, the question is what quantity can play the role of the particle's velocity \mathbf{u} . We have two criteria to meet: (a) the new quantity must reduce to \mathbf{u} when u is small compared with c ; (b) when one changes frames of reference the new quantity must change in a manner that has a simplicity comparable to the nonrelativistic rule $\mathbf{u}' = \mathbf{u} - \mathbf{v}$, if we are to have a hope of conserving momentum in all frames of reference. The velocity \mathbf{u} itself will not do, for under a change of frame of reference \mathbf{u} changes by the relativistic law:²²

$$\mathbf{u}' = \frac{\mathbf{u} - \mathbf{v}}{1 - \mathbf{u}\mathbf{v}/c^2}. \quad (18)$$

It is the denominator in (18) that keeps the transformed total momentum $\mathbf{P}' = m_1\mathbf{u}'_1 + m_2\mathbf{u}'_2$ from having a form simple enough to ensure momentum conservation in the new frame.²³

Why does relativity introduce a complicated denominator into (18)? Think back to the definition of velocity: distance travelled divided by the time it takes. In the nonrelativistic case changing frames changes the distance traveled, *but does not change the time it takes*, so only the numerator changes. In the relativistic case *both* quantities change when you change frames, resulting in the more elaborate rule (18). This suggests an extremely simple and ingenious way out of the problem:

What we should do is generalize the notion of velocity to something which is the distance traveled divided by a time that does *not* depend on frame of reference. Suppose we measure the distance travelled in the usual frame-dependent way, but measure the time that it takes *in a special frame that all observers can agree on*. What could that special

²² The negative sign appears (twice) in (18) because it relates \mathbf{u}' , the velocity of the particle in the new frame, to \mathbf{u} , the velocity of the particle in the old frame, and $-\mathbf{v}$, the velocity of the old frame in the new frame. (The velocity \mathbf{v} is the velocity of the new frame in the old frame.)

²³ The problem is that if we continue to define momentum by (14) but use the relativistic transformation law (18), then the total momentum in the new frame of reference depends in detail on the individual velocities of the particles in the old frame, instead of depending on those velocities only through that particular combination of velocities which is nothing but the total momentum in the old frame, as is the case in the non-relativistic relation (15).

frame be? To ask the question is to answer it: the situation itself singles out one and only one special frame—the *frame of reference in which the particle is at rest*.

Suppose we define a generalized velocity \mathbf{w} to be the distance a particle travels in a given time with the proviso that *this time should always be measured by a clock travelling with the particle*. Note at once that \mathbf{w} reduces back to the ordinary velocity \mathbf{u} when the speed of the particle is small compared with the speed of light, since a clock moving with the particle then runs slowly by an imperceptibly small amount. Now, however, as we go from one frame to another only the distance going into the definition of \mathbf{w} changes, but not the time. So if we redefine the momentum \mathbf{p} to be $m\mathbf{w}$, we might hope to find a simple transformation rule for \mathbf{p} . Since m is invariant, we must inquire how \mathbf{w} transforms.

Now \mathbf{w} is defined in exactly the same way as \mathbf{u} except that the motion of the particle is timed by a clock that is not stationary, but moving with the particle. Such a clock runs *slowly*, so compared with stationary clocks it will indicate that the particle took *less* time to cover a given distance. The slowing down factor $\sqrt{1 - u^2/c^2}$ gives the reduction in time, and therefore \mathbf{w} is *bigger* than the ordinary velocity \mathbf{u} by precisely the factor $1/\sqrt{1 - u^2/c^2}$:

$$\mathbf{w} = \mathbf{u}/\sqrt{1 - u^2/c^2}. \quad (19)$$

With the definition (19) at hand, we can use the transformation rule (18) to find how \mathbf{w} changes when we change to frame moving with velocity \mathbf{v} . In the new frame \mathbf{w}' is given by

$$\mathbf{w}' = \mathbf{u}'/\sqrt{1 - u'^2/c^2}, \quad (20)$$

where \mathbf{u}' is related to \mathbf{u} by the velocity addition law (18). If you substitute (18) into (20) and simplify the resulting expression²⁴ you will find that

$$\mathbf{w}' = \frac{\mathbf{u} - \mathbf{v}}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}}. \quad (21)$$

If we now define relativistic momentum by

$$\mathbf{p} = m\mathbf{w} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}, \quad (22)$$

²⁴ Checking this result is the only slightly messy piece of algebra in this whole business, but the conclusions it leads to are so profound that everybody should suffer through it at least once in a lifetime. If you have not the stomach to do the algebra, at least note that (21) is obviously correct when $\mathbf{v} = 0$ (in which case it reduces to $\mathbf{w}' = \mathbf{w}$), when $\mathbf{v} = \mathbf{u}$ (in which case we have gone to a frame in which the velocity of the particle is 0), and when $\mathbf{u} = 0$ (in which case the particle was originally at rest and therefore has velocity $-\mathbf{v}$ in the new frame.)

then (21) tells us that

$$\mathbf{p}' = \frac{\mathbf{P} - p_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}}, \quad (23)$$

where I have defined a new quantity p_0 by

$$p_0 = \frac{m}{\sqrt{1 - u^2/c^2}}. \quad (24)$$

This is close to what we want, for the momentum in the new frame is now very simply²⁵ related to the momentum in the old frame. The only problem is that something new has appeared as well, p_0 . To see what this might signify, let us first consider what happens when the speed u of the particle is small compared with the speed of light. In that case (24) tells us that p_0 is indistinguishably different from the mass m . If we replace p_0 by m in the transformation law (23), and apply it to the total momentum of a pair of particles, we get

$$\mathbf{P}' = \frac{\mathbf{P} - M\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \quad (25)$$

which except for the denominator is just the familiar nonrelativistic transformation law. The denominator is harmless, however, since it is just a fixed number, that remains the same before and after the collision, and we can conclude from (25) just as we did in the nonrelativistic case, that if \mathbf{P} is the same before and after a collision then \mathbf{P}' will be too, provided the total mass M is conserved in the collision.

But if a particle is not moving at a speed small compared with c then p_0 is *not* its mass m . If we want to insure that momentum, as defined by (22) is conserved in all frames of reference, then we must then replace the law of conservation of total mass by a new law of conservation of total p_0 . Such a replacement is in keeping with the spirit of our attempted generalization of the nonrelativistic conservation laws, for total p_0 is given by

$$P_0 = p_0^1 + p_0^2 = \frac{m_1}{\sqrt{1 - \mathbf{u}_1^2/c^2}} + \frac{m_2}{\sqrt{1 - \mathbf{u}_2^2/c^2}}. \quad (26)$$

Since this reduces to total mass when both velocities are small compared with c , we are discovering that the nonrelativistic law of mass conservation is also a limiting case of a more general relativistic law, just as the nonrelativistic law of conservation of total $m\mathbf{u}$ is a limiting case of conservation of a more general relativistic concept of momentum.

²⁵ The factor $\sqrt{1 - v^2/c^2}$ in the denominator in (23) may not strike you as so simple, but remember that it is only a number, determined by the relative velocity of the two frames. It is independent of the speed of the particle itself, and therefore exactly the same number will appear in the rule that gives the momentum in the new frame of every particle participating in the collision.

But before we can declare there to be a new conservation law for P_0 , we must check to see whether it too passes the crucial requirement that a genuine law must hold in all frames of reference. This leads us to one more unpleasant computation very much like to the one that led us to (23).²⁶ We must apply (18) to the definition

$$p'_0 = \frac{m}{\sqrt{1 - u'^2/c^2}} \quad (27)$$

to express p'_0 in terms of quantities in the original frame. When this is done we find:

$$p'_0 = \frac{p_0 - \mathbf{p}\mathbf{v}/c^2}{\sqrt{1 - v^2/c^2}}. \quad (28)$$

This has a structure very similar to the transformation rule (23) for momentum. Because both structures are so simple, the transformations (23) and (28) for the individual particle \mathbf{p} and p_0 lead to transformations for the total momentum \mathbf{P} and total P_0 of *exactly* the same forms as (23) and (28) :

$$\mathbf{P}' = \frac{\mathbf{P} - P_0\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \quad (29)$$

$$P'_0 = \frac{P_0 - \mathbf{P}\mathbf{v}/c^2}{\sqrt{1 - v^2/c^2}}. \quad (30)$$

Since these express \mathbf{P}' and P'_0 entirely in terms of \mathbf{P} and P_0 (and the relative velocity \mathbf{v} of the two frames) *if the unprimed quantities are the same before and after a collision, the primed quantities must be too*. Therefore if \mathbf{P} and P_0 are both conserved in one frame they will both be conserved in any other frame. Our proposed relativistic generalization (22) of the definition of momentum meets all of our criteria for a conserved quantity, as does the new quantity P_0 whose conservation we are considering.

What are the implications of replacing the nonrelativistic conservation of total mass M by the relativistic conservation of P_0 ? How are we to interpret p_0 and the sum P_0 of the values of p_0 for a group of particles? We can get a powerful clue by examining the structure of p_0 for a particle whose speed u is small compared with c . In this limit the definition (24) merely tells us what we already know: that p_0 is very close to m , the mass of the particle. But since we are trying to make sense of the difference between the old nonrelativistic law of conservation of M and a new relativistic law of conservation of P_0 , what we really require is an estimate of the *difference* between p_0 and m when u is small compared with c . We would like that estimate to be a little more informative than the

²⁶ You can extract the result (28) more deftly by dividing the left side of the momentum transformation law (23) by the left side of the velocity transformation law (18) and the right side by the right side, and comparing what you get with the definitions of p_0 and \mathbf{p} .

simple declaration that the difference is very small. In the Appendix A at the end of these notes we construct such an estimate,²⁷ showing that when u is very small compared to c , then to a high degree of accuracy,

$$p_0 - m = \frac{1}{2}mu^2/c^2. \quad (31)$$

Thus at nonrelativistic velocities $p_0 - m$ is *nothing but the nonrelativistic kinetic energy divided by c^2* .

So if we define the *relativistic kinetic energy* by

$$k = p_0c^2 - mc^2, \quad (32)$$

then k does indeed reduce to the ordinary nonrelativistic kinetic energy at speeds small compared to c and we have our interpretation of p_0 : the interesting quantity is not p_0 itself, but the product of p_0 with c^2 , which is the sum of two terms:

$$p_0c^2 = mc^2 + k. \quad (33)$$

We have now reached our goal. In order for the relativistic momentum \mathbf{P} to be conserved it is necessary for P_0 to be conserved as well. But

$$P_0c^2 = Mc^2 + K, \quad (34)$$

where M is the total mass and K , the total kinetic energy.

Recall now the nonrelativistic state of affairs. Total mass M is always conserved, but total kinetic energy K is only conserved in elastic collisions. Relativistically we can continue to define elastic collisions as those in which K is conserved. But relativistically P_0 must *always* be conserved.²⁸ Since P_0 is related to M and K through (34), it follows that if K is conserved then M must be conserved as well. But if K is not conserved, then M cannot be conserved either. In an inelastic collision if the total kinetic energy goes down (or up) by²⁹ ΔK then in order for P_0 to be conserved in the collision, (34) requires that *the change in kinetic energy must be precisely balanced by an increase (or decrease) in the total mass by ΔM* , where

$$\Delta Mc^2 = \Delta K. \quad (35)$$

²⁷ The analysis is too simple to require an Appendix, but I can't stand to interrupt the narrative at this exciting moment.

²⁸ For if it were not, remember, momentum could not be conserved in all frames.

²⁹ By ΔK we just mean the change in K as a result of the collision: $\Delta K = K^a - K^b$, and similarly, $\Delta M = M^a - M^b$.

This must be true whether the collision involves relativistic or nonrelativistic velocities, since the relativistic theory ought to be valid for all velocities. Why, then did we never notice it in collisions at nonrelativistic velocities, where total mass appeared to be conserved? The reason is that the change in mass is then extremely small. This change is $\Delta M = \Delta K/c^2$, and a measure of the size of ΔK , the change of kinetic energy, is the total mass M times the square of a typical particle velocity u^2 . Thus the change in ΔM is typically the mass M itself times a factor whose size is roughly u^2/c^2 . At less than supersonic velocities, u^2/c^2 is less than 1/1,000,000,000,000.

So the change in mass required in inelastic collisions by the relativistic theory is too small to be noticed in nonrelativistic collisions. The exact relativistic conservation of P_0c^2 simply masquerades as conservation of total mass when all speeds are small compared with c . But at relativistic speeds the consequences of the correct relativistic conservation law can be profound.

Returning from the sublime to the merely conventional, I note that one defines P_0c^2 to be E , the total energy, and defines \mathbf{p}_0c^2 for each individual particle to be its energy e . One then has the energy and momentum of a particle of mass m and velocity u defined by

$$e = \frac{mc^2}{\sqrt{1 - u^2/c^2}}, \quad (36)$$

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}. \quad (37)$$

The rules (29) and (30) become

$$E' = \frac{E - \mathbf{P}\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \quad (38)$$

$$\mathbf{P}' = \frac{\mathbf{P} - E\mathbf{v}/c^2}{\sqrt{1 - v^2/c^2}}. \quad (39)$$

Note that (36) asserts that the energy e of a particle of mass m has the value mc^2 when the particle is at rest. This is sometimes incorrectly cited to be the meaning of $E = Mc^2$. But this by itself is merely an uninteresting matter of convention. One could equally well have defined the energy of a particle to be its kinetic energy, $k = e - mc^2$, in which case the energy of a particle at rest would be zero. The true meaning of $E = Mc^2$ is to be found in the study of inelastic collisions, as the expression (35) of the balance between changes in total kinetic energy and changes in total mass.

Thus if two particles collide in their zero-momentum frame and stick together to form a final particle at rest, the mass of that final particle will exceed the sum of the masses

of the two incident particles by precisely their kinetic energy prior to the collision divided by c^2 . Conversely, if a particle at rest spontaneously disintegrates into two particles that go flying off, the total mass of the two particles is less than the mass of their parent by precisely their kinetic energy divided by c^2 .

If you wish to create new particles, more massive than any that have been observed to date, it is necessary to fling together less massive particles at speeds close to c to provide the kinetic energy needed to supply the additional post-collision mass. This is a matter that was of considerable interest to the Congress of the United States and the economy of the state of Texas in the early 1990's. Less expensive and still highly viable versions of the same process take place on the Cornell campus under Upper Alumni Field.³⁰

Photons

Note that (36) implies that the energy of a particle becomes arbitrarily large as its speed approaches that of light—yet another illustration of the difficulty of accelerating anything up to the speed of light. Yet there are particles (the photon, for example) that do move at the speed of light. How are we to account for this? Evidently (36) allows a particle to move at a speed u equal to the speed c of light without requiring an infinite amount of energy to do so, provided the mass of such a particle is zero. At first glance it appears that (36) and (37) can tell us nothing useful about zero mass particles with speeds $u = c$, since dividing 0 by 0 is a famous way of arriving at utter nonsense. But in fact there are two consequences of these two equations that remain perfectly well defined in the limit of zero m .

It follows from (36) and (37) that

$$e^2 = p^2 c^2 + m^2 c^4. \quad (40)$$

and that

$$\mathbf{p} = e\mathbf{u}/c^2. \quad (41)$$

Indeed Eqs. (41) and (40) are completely equivalent to (36) and (37),³¹ but they have the virtue of retaining an intelligible content even when applied to particles of zero mass. When $m = 0$ (40) reduces to

$$p = e/c \quad (42).$$

³⁰ Cornell has been for many years the last university in the United States where such experiments continue to be done under the direct management of the Physics Department.

³¹ That is to say, you can reverse the process. Starting with (41) and (40) you can deduce (36) and (37). It turns out that for most purposes (41) and (40) are much easier to work with than (36) and (37) so that while the latter pair of equations play a fundamental role in motivating the new definitions of energy and momentum, it is the former pair that capture their most important features.

This is consistent with (41) provided the speed u of the zero mass particle is equal to the invariant speed c . Thus the relativistic definitions of energy and momentum apply perfectly well to a particle of zero mass, where they reduce (a) to the requirement that the speed of such a particle is necessarily c , and (b) to the condition that the energy of such a particle is just c times the magnitude of its momentum. This turns out to be extremely useful. For an illustration, see Appendix B.

How fast does something move through time?

Note, finally, a way of viewing the quantity p_0 from a somewhat different perspective, which ties together the concepts of energy and momentum in a way that is simply unavailable in the nonrelativistic case. The momentum of a particle in any given frame of reference is the product of the mass of the particle with the rate at which the particle moves through *space* as measured by a clock moving with the particle. In quite the same way p_0 is the mass of the particle times the rate at which the particle moves through *time*, as measured by a clock moving with the particle.

To nonrelativistic ears this sounds crazy: how can something move through time at anything but a rate of one second per second.³² But relativistically it makes perfect sense as a rather elegant way to express the slowing down of a moving clock. The higher the speed of a particle (in a given frame of reference) the more rapidly the particle moves through time (as time is measured in that frame of reference) according to a clock moving with the particle (which measures time in the proper frame of the particle). Thus in a frame in which a particle moves at $\frac{3}{5}$ the speed of light, it moves through time at a rate of $\frac{5}{4}$ of a second per proper second. This is just a dramatic, upside-down, and in some deep sense more meaningful way of saying that any internal clock-like processes associated with the particle run slowly by the appropriate slowing down factor: for every second that passes on any clock moving with the particle, time in the frame in which we are describing this motion advances by 1.25 seconds.

When something speeds up its passage through space, so that it takes it less proper time to get from here to there, it also speeds up its passage through time, so that it takes it less proper time to get from now to then.

Appendix A: A convenient form for $p_0 - m$.

It follows from the definitions of \mathbf{p} and p_0 (22) and (24) that

$$p_0^2 - \mathbf{p}^2/c^2 = m^2 \tag{43}$$

³² And indeed, non-relativistically conservation of p_0 is just conservation of mass.

or

$$p_0^2 - m^2 = (p_0 - m)(p_0 + m) = \mathbf{p}^2/c^2 \quad (44)$$

or

$$(p_0 - m) = \frac{\mathbf{p}^2}{(p_0 + m)c^2}. \quad (45)$$

The left side of (45) is what we are looking for: the difference between p_0 and m . The right side unfortunately contains p_0 again, but if we are only interested in speeds u small compared with c , then p_0 is exceedingly close to m . Consequently when u is small compared with c we can evaluate the right side of (45) with very high accuracy if we replace the p_0 on the right simply with m . Under these same conditions \mathbf{p} is also very close to the non-relativistic value $m\mathbf{u}$. Making both these replacements on the right side of (45) gives us the estimate we are looking for. When a particle moves slowly compared with the speed of light, to a high degree of precision,

$$p_0 - m = \frac{1}{2}mu^2/c^2. \quad (46)$$

Appendix B:

What happens when a photon collides with a stationary particle.

As a simple illustration of how the relativistic conservation laws work in an extreme relativistic case, consider a collision between a photon (which of course moves at the extremely relativistic speed c) and an initially stationary particle of mass m_i , in which the photon is absorbed by the particle.³³ If the photon has energy ω (“omega”, a popular notational choice for the energy of a photon) how fast does the particle move after it has absorbed the photon, and what is the particle’s new mass m_f ? (The subscripts i and f stand for “initial” and “final”.)

The answers fall directly out of the conservation laws for total energy and momentum:

Energy conservation. Before the collision the photon has energy ω and the particle has energy $m_i c^2$ (since this is what (36) — or (40) and (41) together — gives for a particle with mass m_i and speed $u = 0$.) After the collision the particle has swallowed up the photon and has energy e . Conservation of energy requires:

$$\omega + m_i c^2 = e. \quad (47)$$

Momentum conservation. Before the collision the photon has momentum k , which (42) tells us is related to its energy ω by

$$k = \omega/c. \quad (48)$$

³³ This is a relativistic version of the collision between the two particles that stick together and form a single compound particle after the collision.

Before the collision the particle has momentum 0, since it is stationary. After the collision it has momentum p and there is no photon left. So conservation of total momentum requires the particle to have all the momentum originally possessed by the photon:

$$\omega/c = p. \quad (49)$$

Now if you know the energy and the momentum of an object, then you can most easily extract its velocity directly from (41): if something moves with velocity \mathbf{u} its energy e and momentum \mathbf{p} are related by $\mathbf{p} = e\mathbf{u}/c^2$. Therefore the ratio of its speed to the speed of light is given by

$$u/c = cp/e. \quad (50)$$

Using the forms (49) and (47) for p and e gives the answer:

$$u/c = \frac{1}{1 + m_i c^2 / \omega}. \quad (51)$$

If $m_i c^2$ is large compared with the energy ω of the photon, then the speed of the particle after the collision is a small fraction of the speed of light. But when the energy ω of the photon becomes comparable to $m_i c^2$ of the particle, the speed with which the particle recoils becomes comparable to c . To get the particle moving at speeds very close to the speed of light c , the energy ω of the photon must become much larger than $m_i c^2$. Note, though, that no matter how large ω becomes, (51) still gives a final speed u for the particle that is less than the speed of light.

The simplest way to get at the mass m_f of the particle after it has absorbed the photon is through the relation (40) between the energy, momentum, and mass of a particle. Applied to the particle after it has absorbed the photon, this gives

$$(m_f c^2)^2 = e^2 - (pc)^2. \quad (52)$$

Using the forms (47) and (49) for e and p we learn from (52) that m_f satisfies

$$(m_f c^2)^2 = (\omega + m_i c^2)^2 - \omega^2 = (m_i c^2)(2\omega + m_i c^2), \quad (53)$$

so the mass of the particle after it has absorbed the photon has become

$$m_f = m_i \sqrt{1 + 2\omega/m_i c^2}. \quad (54)$$

Thus the mass m_f of the particle after it has absorbed the photon can be significantly larger than its original mass m_i , provided the energy ω of the photon is comparable to or exceeds $m_i c^2$.

Note that one can use (51) to reexpress the relation (54) between the initial and final masses in terms of the velocity u of the final particle. The result is the curious fact the ratio of the masses is given by nothing but the Doppler shift factor:

$$m_f/m_i = \sqrt{\frac{1 + u/c}{1 - u/c}}. \quad (55)$$

Appendix C: A summary in the form of a table:

Conservation of momentum, energy, and mass
relativistically and nonrelativistically

	Non-Relativistic	Relativistic
MASS	$M = m_1 + m_2$	$M = m_1 + m_2$
<i>Conserved?</i>	<i>always</i>	<i>elastic collisions only</i>
<i>Transformation</i>	$M' = M$	$M' = M$
MOMENTUM	$\mathbf{P} = m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2$	$\mathbf{P} = \frac{m_1 \mathbf{u}_1}{\sqrt{1 - \mathbf{u}_1^2/c^2}} + \frac{m_2 \mathbf{u}_2}{\sqrt{1 - \mathbf{u}_2^2/c^2}}$
<i>Conserved?</i>	<i>always</i>	<i>always</i>
<i>Transformation</i>	$\mathbf{P}' = \mathbf{P} - M\mathbf{v}$	$\mathbf{P}' = \frac{\mathbf{P} - \mathbf{v}E/c^2}{\sqrt{1 - \mathbf{v}^2/c^2}}$
ENERGY	$E = \frac{1}{2}m_1 \mathbf{u}_1^2 + \frac{1}{2}m_2 \mathbf{u}_2^2$	$E = \frac{m_1 c^2}{\sqrt{1 - \mathbf{u}_1^2/c^2}} + \frac{m_2 c^2}{\sqrt{1 - \mathbf{u}_2^2/c^2}}$
<i>Conserved?</i>	<i>elastic collisions only</i>	<i>always</i>
<i>Transformation</i>	$E' = E - \mathbf{P}\mathbf{v} + \frac{1}{2}M\mathbf{v}^2$	$E' = \frac{E - \mathbf{v}\mathbf{P}}{\sqrt{1 - \mathbf{v}^2/c^2}}$

Comments on the Table

1. “Conserved” means that the quantity is the same before and after the collision.
2. The entries under “Transformation” give with a prime (') the value the quantities have in a frame moving with velocity \mathbf{v} with respect to a frame in which they have values without primes.
3. The same relations hold for any number of particles. The number of particles before and after the collision need not be the same. If there is only one particle before (or after) the “collision” then we are describing a particle that breaks up into more than one (or several particles fusing into one).
4. When the speed of a particle is small compared with the speed of light then its energy, $\frac{mc^2}{\sqrt{1-\mathbf{u}^2/c^2}}$ is very nearly equal to $mc^2 + \frac{1}{2}m\mathbf{u}^2$.
5. In any frame of reference u is equal to the distance a uniformly moving particle goes divided by the time it takes it to go that distance; $\frac{u}{\sqrt{1-\mathbf{u}^2/c^2}}$ is equal to that same distance divided, now, by the time it takes the particle to advance that much distance, as measured by a clock moving with the particle.
6. The conservation laws obey the principle of relativity: if they hold in one inertial frame then they hold in all inertial frames. (This is true for the nonrelativistic quantities only if one uses the nonrelativistic rules for changing frames of reference.)
7. Note the different roles played by inelastic collisions in the relativistic and nonrelativistic theories. Non-relativistically mass is conserved even in inelastic collisions but kinetic energy is not; relativistically energy is conserved even in inelastic collisions but mass is not.