10. Space-Time Diagrams

We have been describing various events taking place along a long straight railroad track (or along a long straight line of rockets) by representing the events as points on a blackboard or a piece of paper. Events taking place in the same place at the same time (coincident events) are represented by the same point.[1] In many of these figures we have taken a horizontal separation of two points to indicate a spatial separation of the events they represent, and a vertical separation to indicate a temporal separation. By exploring this kind of procedure a little more generally and systematically, it is possible to arrive at deeper — I would say, in fact, the deepest — understanding of what relativity has to tell us about the nature of space and time.

For simplicity we continue to deal with only one spatial dimension — all the events we shall consider take place along a single straight track.[2] Let us start with a particular frame of reference (Alice's) and specify some simple rules that Alice can use to specify events by points on a page. Until Bob appears on the scene everything that follows refers to Alice's frame of reference. When I talk about events happening in the same place (or at the same time) I mean at the same place (or at the same time) according to Alice.

Rule 1. Two or more events that happen at the same place *and* at the same time (space-time coincidences) are all represented by the same point.

Rule 2. Events happening at the same place (but not necessarily at the same time) are represented by points on a single straight line. (Figure 1.) The line is called a line of constant (or fixed) position (or place).[3] Alice is clearly free to orient one such line of constant position in any direction she chooses, since such a choice amounts to nothing more than appropriately orienting the page on which she draws her diagram.

Rule 3. Any two such lines of constant position representing various events that happen in two *different* places must be parallel. For if they were not parallel they would intersect somewhere, and their point of intersection would correspond to a single event that happened in two different places. But by definition an event is something that happens at a single place (and at a single time).

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[1] If we wish to draw a picture representing an event we can't of course, make the picture as small as a geometric point; similarly if we wish to draw two pictures representing both of two coincident events, we try to make the pictures as close together as we can.

[2] Adding the other two dimensions — horizontal and vertical distance away from the track — can sometimes give further insight, but it makes it impossible to draw everything on a page or blackboard. We shall therefore continue to restrict our attention to a single spatial dimension.

[3] If you want a more compact term try *equiloc* or *isotop*. As far as I know neither term is actually used by anybody, but one of them should be.

Rule 4. In analogy with the usual conventions of map makers (more precisely, those making maps of regions very small compared with the radius of the Earth) Alice takes the distance on the page between two distinct lines of constant position to be proportional to the actual distance between the positions of the events they represent. The quantitative relation between distances in space and distances in the diagram is given by a scale–factor λ ("lambda"). Multiplication by λ converts the actual spatial distance between two events into the distance on the page between the lines of constant position on which the events lie in the diagram. For example if lines of constant position separated by one cm on the page corresponded to events at positions one km apart, then λ would be one centimeter to the kilometer, numerically 1/100,000. If we wish to distinguish Alice's scale factor from those of people using other frames of reference (and it turns out to be important to be able to do this) we can give it a subscript, calling it λ_A.

The next three rules (5–7) simply specify for location in time, what Rules 2–4 specify for location in space.

Rule 5. Events happening at the same time (but not necessarily in the same place) are represented by points on a single straight line. (Figure 2.) The line is called a line of constant (or fixed) time.[4] Alice is free to orient one such line of constant time to make any angle she wishes with her lines of constant position (except $0°$, as noted in Rule 8 below), since such a choice of direction amounts to nothing more than an appropriate stretching of the page on which she draws her diagram (considered only for this purpose to be made of rubber).

Rule 6. Any two different lines of constant time must be parallel. For if they were not parallel they would intersect somewhere, and their point of intersection would correspond to a single event happening at two different times. But by definition an event is something that happens at a single time (and at a single place).

Rule 7. Alice takes the distance on the page between two distinct lines of constant time to be proportional to the actual time interval between the times of the events they represent. We defer her choice of scale factor to Rule 9.

The next three rules (8–10) deal with the relation between lines of constant position and lines of constant time.

Rule 8. Any line of constant time intersects any line of constant position in precisely one point, which represents those events that happen precisely at *that* time and in *that* place. Consequently the common direction of all Alice's lines of constant time, though it is otherwise hers to choose, cannot be the same as the common direction of all her lines of constant position. The two families of lines must cross at some non-zero angle θ ("theta").

Rule 9. It turns out to be extremely convenient for Alice to take the distance in the diagram between two lines of constant time representing events one nanosecond apart to

[4] One could also call it an *equitemp* or an *isochron*. Like *equiloc* and *isotop*, these useful terms have not entered the standard lexicon of relativity.

be exactly the same as the distance in the diagram between two lines of constant position representing events one foot[5] apart. Putting it in terms of scale factors, the scale factor λ for lines of constant position (in centimeters of diagram per f) is numerically the same as the scale factor λ for lines of constant time (in centimeters of diagram per ns.)

Rule 10. With the convention adopted in 9, it follows that another convenient scale factor, the distance μ ("mu") along any line of constant position associated with two events one ns apart, is also exactly the same as the distance along any line of constant time associated with two events one f apart. (See Figure 3.) This is a consequence of the elementary fact that when a pair of parallel lines intersects another pair of parallel lines separated by the same distance as the first pair (in this case the distance is just λ), then the parallelogram defined by the four points of intersection has four equal sides.[6] Note that the scale factor μ exceeds the scale factor λ unless Alice takes her lines of constant time perpendicular to her lines of constant position, in which case $\mu = \lambda$.

Both scale factors are useful. Often it is easiest to extract the time (or distance) between events from the distance between the lines of constant time (or position) on which they lie, in which case λ is the relevant scale factor. But sometimes one wants to extract the time (or distance) between events that happen in the same place (or at the same time) from their distance apart on a line of constant position (or time), in which case μ is the relevant one.

A particularly important collection of events for an object small enough to be considered to occupy just a single point of space at any moment of time, is the set of *all* events at which the object is present. The totality of all such events is represented by a continuous line in the diagram. This line, which represents the entire history of the object, is called the *world line* or *space-time trajectory* of the object. For example an object stationary in Alice's frame of reference throughout its entire history is represented by the line of constant position associated with the place the object occupies. An object moving uniformly in Alice's frame of reference is represented by a straight line that is not parallel to any line of constant position, since the object is at different positions at different times. An object that is moving non-uniformly — for example back and forth — is represented by a wiggly line.

Objects moving uniformly with the same velocity make parallel lines. This is because in the same time, according to Alice, they cover the same distance in the same direction. Alternatively, if their space-time trajectories were not parallel they would meet in a point. That point would represent an event in which the objects were in the same place at the same time. But if their velocities are the same they would then have to be in the same

[5] Recall that the foot is defined in Physics 209 to be the distance light travels in vacuum in one nanosecond.

[6] A parallelogram with four equal sides is called a rhombus. A rhombus has the elementary property that the lines connecting opposite vertices — called "diagonals" — bisect the angles at those vertices. This will turn out to be useful.

place at all times and their trajectories would be identical. The trajectories of objects moving with the same speed in opposite directions (and therefore with different velocities) are not, of course, parallel. Such a pair of objects can be in the same place at just a single moment.

An especially important world line is the space-time trajectory of a photon, or of any other object moving at the speed of one foot per nanosecond. The final three rules (11–13) have to do with photon trajectories.

Lines of constant position and constant time have a very simple relation to photon trajectories. Any two events on a photon trajectory must be as many feet apart in space as they are nanoseconds apart in time. So as a result of Rule 9, the two lines of constant position passing through those two events must be the same distance apart in the diagram as the two lines of constant time passing through those events. Since the photon trajectory is thus the diagonal of a rhombus formed by the two pairs of parallel lines, the trajectory bisects the angles at the two vertices it connects. (See Figure 4.) We have thus deduced an extremely important property of Alice's diagram:

Rule 11. The angle that lines of constant position make with the trajectory of a photon must be the same as the angle that lines of constant time make with that trajectory. Putting it another way, *the two photon trajectories through the point of intersection of a line of constant position with a line of constant time, bisect the angles formed by the two lines.*

Since this rule applies to photons moving in either direction we have a second important deduction (which follows from Rule 11 for the reasons shown in Figure 5):

Rule 12. The trajectories of two photons moving in opposite directions are *perpendicular* to each other in the diagram. Therefore even though Alice can freely chose the angle θ between her lines of constant time and constant position, the scale convention adopted in Rule 9 requires certain angles to be fixed: *the world lines of oppositely moving photons are necessarily perpendicular.*

Rule 13. Because of Rules 11 and 12, Alice can rotate her page so that the trajectories of two photons moving in opposite directions are symmetrically disposed about the vertical direction, tilted at 45 degrees to the right and left, with the times of the events on each photon trajectory increasing as one moves along the page in the upward direction. Because Alice's lines of constant time make the same angle with the photon trajectories as her lines of constant position, her lines of constant position will then tilt away from the vertical at the same angle that her lines of constant time tilt away from the horizontal. It is conventional always to orient a space-time diagram in this way, so that the vertical and horizontal directions bisect the right angles between the two families of photon trajectories, and so that lines of constant time higher in the diagram represent events occurring at later times.

With this convention for orienting the diagram, lines of constant time are always more vertical than horizontal (i.e. their angle with the vertical is less than 45°), while lines of constant position are always more horizontal than vertical (i.e. their angle with the horizontal is less than 45°).

The above rules completely determine the structure and orientation of the system of lines of constant time and position that Alice uses to locate events in space and time except for two choices still available to her:

(a) She is free to choose the scale factor λ — i.e. the distance on the page between two lines of constant position associated with places one f apart (which is also the distance on the page between two lines of constant time associated with events one ns apart.)

(b) She is free to choose the angle θ that her lines of constant time make with her lines of constant position or, equivalently, the angle $\frac{1}{2}\theta$ that both families of lines make with the photon lines. (Her choice of λ and θ together fix the alternative scale–factor μ.)

Alice's choice of scale depends, of course, on how big a page she has and on the spatio–temporal extent of the collection of events she wishes to represent in her diagram. Her choice of angle depends on what she (or we) wish to do with her diagram. If she is using it only for her own private purposes then a pleasingly symmetric choice is to take θ to be 90 degrees, so that her lines of constant position are vertical and her lines of constant time, horizontal. If, however, she (or we) wish to compare the space-time description of events that she reads from her diagram with the space-time description of those same events provided by other observers using one or more other frames of reference, then taking θ to be 90 degrees need not give the clearest picture. To see why we must consider the use to which Alice's diagram can be put by people who prefer to describe events using other frames of reference.

Bob, moving uniformly along the track with velocity v with respect to Alice, wishes to describe the same events that she has been describing, but prefers a frame of reference in which he is at rest. Suppose Bob is shown Alice's diagram, filled with points that represent isolated events and lines that represent space-time trajectories, but without any of her lines of constant time and position that she might have drawn to help her locate those events in space and time. Rather than make his own independent diagram to describe those various phenomena, Bob can use precisely the same collection of points and lines that Alice used. But he will describe them in a different spatio–temporal language, since he will disagree with Alice's general notions of "same place" and "same time". He will therefore not use the same lines of constant position and time that Alice uses. It is not hard to figure out what he must do.

If Bob's frame of reference has velocity v with respect to Alice's, then Bob's lines of constant position must be parallel to the space-time trajectory of a particle that Alice maintains is moving with velocity v. Thus Bob's lines of constant position are parallel straight lines that are not parallel to Alice's lines of constant position. The faster Bob moves with respect to Alice, the more they tilt away from Alice's lines. Lines of constant

Alice-time and Alice-position through any two points on one of Bob's lines of constant position, define a parallelogram the ratio of whose sides (or the ratio of the distances between whose sides) is just the velocity v of his frame with respect to hers. This is illustrated in Figure 6.

We can also determine the orientation of Bob's lines of constant time. We do this by putting into the diagram a set of events associated with a clock synchronization experiment, carried out on a train that is stationary in Bob's frame of reference. The left end, right end, and middle of such a train are represented in Alice's diagram by lines of constant Bob-position. Since Alice agrees with Bob about what point on the train constitutes its middle, the lines are equally spaced in Alice's diagram. Two photons created together in the middle of the train travel in opposite directions at the same speed. Since the train is stationary in Bob's frame, and both photons have the same speed in his frame (namely one f/ns — this is the first place where the invariance of the velocity of light enters our story) they arrive at the two ends of the train at the same Bob-time. So if we draw a pair of 45° lines (recall Alice's Rule 13 above) that start at a point on the trajectory of the middle of the train, representing the trajectories of photons moving toward the front and rear, then the points of intersection of the two photons with the two ends of the train represent simultaneous events in Bob's frame and therefore lie on one of his lines of constant time. (All this is illustrated in part (a) of Figure 7.)

It is then easy to deduce that Bob's lines of constant time in Alice's diagram must make the same angle with the photon trajectories as his lines of constant position do. One sees this most directly by letting each photon be reflected from its end of the train back to the middle. The resulting collection of photon trajectories (shown in part (b) of Figure 7) form the four sides of a rectangle. It is evident (as explained in part (b) of the caption of Figure 7) that all the angles with the same label are equal. Therefore the two photon trajectories passing through the black dot on the left do indeed bisect the angles between Bob's lines of constant position and time passing through that dot.

Note that this conclusion is identical to Rule 11 for the orientation of Alice's lines of constant time and position. Furthermore, because the common speed of both photons in Bob's frame continues to be 1 f/ns, two of Bob's lines of constant position associated with places one f apart in his frame, must be the same distance apart in Alice's diagram as two of his lines of constant time associated with times one ns apart in his frame.

Thus the rules we set up for the orientation of Alice's lines of constant time and constant position and the relation between their scales, impose restrictions on the lines of constant position and time that Bob must use, if he wishes to represent events with the same points that Alice uses in her diagram. Importantly, those restrictions turn out to have exactly the same form as the rules we originally imposed on Alice. It is therefore impossible for anybody else to tell which of them made the diagram first, following Rules 1-13, and which of them subsequently imposed his or her own lines of constant time and position on the other's diagram. This wonderful symmetry is required by the principle of relativity. Seeing it emerge in this way affords a vivid demonstration that the principle of relativity is indeed consistent with the frame independence of the velocity of light.

The fact that Bob's and Alice's lines of constant time and position are both symmetrically situated about the 45° photon lines, has as an immediate consequence the $T = Dv/c^2$ rule for simultaneous events, in the form $T = Dv$ that the rule assumes when one measures times in nanoseconds and distances in feet. This is spelled out in Figure 7a.

In summary, Alice and Bob (and Carol and Dick and Eve...) can all represent events in space and time by the same set of points in a single diagram, on which they each superimpose different families of lines of constant time and position. The lines in any one observer's family are symmetrically disposed about the two perpendicular directions along which photon trajectories lie — i.e. the photon trajectories passing through points of intersection of lines of constant time and position belonging to any single one of the various frames of reference, bisect the angles between those two lines.

There remains the question of how people using different frames of reference relate their scale factors λ which give the distance on the page between their lines of constant position associated with events one f apart and between their lines of constant time associated with events one ns apart. One can acquire substantial insight from appropriately drawn space-time diagrams without ever needing the quantitative relation between scale factors, so I simply state here what the rule is:[7]

Call a rhombus bounded by lines of constant time and position associated with events one ns and one f apart a *unit rhombus*. The scale factors used by different frames of reference are related by the rule[8] that *unit rhombi used by different observers all have the same area*. Since the altitude of a unit rhombus is the scale factor λ and its base is the scale factor μ (Figure 8), the analytical expression of this geometric rule is that for any two frames of reference

$$\lambda_A \mu_A = \lambda_B \mu_B. \tag{10.1}$$

Figures 9-11 show a few ways in which these space-time diagrams clarify some of the puzzles raised by relativity:

(1) Figure 9 shows how it is possible for each of two sticks in relative motion to be longer than the other in its proper frame. The two solid vertical lines represent the space-time trajectories of the left and right ends of the first stick. Lines of constant position in the proper frame of the first stick are vertical (since each end of the stick does not change

[7] No use is made of this rule in the examples that follow.

[8] If you are curious about where this rule comes from, see the Appendix below, where I show that this rule follows directly from the requirement that when Alice and Bob move away from each other at constant velocity each must *see* the other's clock running equally slowly.

its position in that frame), so according to Rule 13, lines of constant time in the proper frame of the first stick must be horizontal. Any horizontal slice of the figure shows what things are like at that given moment of time in the frame of the first stick.

The two parallel solid vertical lines that slant upward to the right represent the space-time trajectories of the left and right ends of the second stick. They constitute lines of constant position in the proper frame of the second stick. Lines of constant time in the proper frame of the second stick tilt away from the horizontal by as much as the lines of constant position tilt away from the vertical. (This is Rule 13 again.) Any slice of the figure with such a tilted line of constant time shows what things are like at a particular moment of time in the frame of the second stick.

The horizontal dashed line in Figure 9 is a line of constant time in the frame of the first stick. As you look along that line from left to right you encounter first the left end of the first stick, then the left end of the second, then the right end of the second, and finally the right end of the first. Thus in the proper frame of the first stick the two ends of the first stick extend beyond the two ends the second stick: the second stick is shorter than the first.

On the other hand the tilted dashed line is a line of constant time in the frame of the second stick. As you move along that tilted line from lower left to upper right you encounter first the left end of the second stick, then the left end of the first, then the right end of the first, and finally the right end of the second. Thus in the proper frame of the second stick the two ends of the second stick extend beyond the two ends of the first stick: the first stick is shorter.

What the figure makes absolutely explicit is that if two sticks are in motion relative to one another, then their comparative lengths depend on the convention one employs for the simultaneity of events in different places. The various pieces of a stick (its two ends, its middle, a point two thirds of the way along the stick, etc.) are situated in different places. Which parts of the space-time trajectories of each piece of the stick one puts together to make up what one would like to call *the stick* at a given moment of time, depends on which events in the history of each of those spatially separated pieces of the stick one chooses to regard as simultaneous. What is independent of any such convention, is the totality of all the space-time trajectories of all the pieces of both sticks. What is conventional and frame-dependent, is how one chooses to slice those trajectories with lines of constant time to form the *stick-at-a-given-moment.*

Note that there is a third frame of reference (moving to the right with respect to the first stick at a speed less than the second) in which both sticks have the same length. That frame is the one which has a line of constant time that can join the point of intersection of the trajectories of the left ends of the sticks with the point of intersection of the trajectories of the right ends.

(2) Figure 10 shows how it is possible for each of two clocks, in relative motion, to run

faster than the other in its proper frame. The vertical row of numbered circles represents seven moments in the history of a clock and the reading of the clock (in seconds) at those moments. The slanting row represents six moments in the history of a second clock, moving to the right relative to the first and its reading (in seconds) at those moments. Both clocks are in the same place at the same time when they read 0, and are therefore represented at that moment in their histories by one and the same circle. Everybody, regardless of what frame of reference they use, agrees that both clocks read zero at the same time, because the clocks read zero at the same time *and* iny the same place.

Lines of constant position in the proper frame of the first clock are vertical (since the line on which the seven moments in the history of the first clock lie is vertical) so lines of constant time in the proper frame of the first clock are horizontal. Since the second clock reading 4 and the first clock reading 5 lie on a single horizontal line, those two readings happen at the same time, according to the proper frame of the first clock. Since both clocks also read 0 at the same time, the second clock is running at 4/5 the rate of the first, according to the proper frame of the first clock.

Lines of constant position in the proper frame of the second clock have the same tilt as the line on which the six moments in the history of the second clock lie. Lines of constant time in the proper frame of the second clock make the same angle with the horizontal as lines of constant position make with the vertical. Such a line of constant time is shown connecting the moment when the first clock reads 4 and the second clock reads 5. Since both clocks also read 0 at the same time, the first clock is running at 4/5 the rate of the second, in the according to the proper frame of the second clock.

Figure 10 makes explicit the fact that a comparison of the rates of two clocks in relative motion depends crucially on the convention one adopts for the simultaneity of events in different places.

(3) If we stopped with Figure 10, which clock was *actually* running slower would be a matter of convention, empty of real content. Suppose, however, that the second clock suddenly reverses its direction of motion and returns to the first. One can then compare them directly when they are back at the same place at the same time and see which has advanced by the greater amount.

In thinking about this it is important to recognize that the process of turning around breaks the symmetry between the two clocks. The first clock is stationary in a single inertial frame of reference throughout its entire history. The proper frame of the second clock, however, changes from one inertial frame of reference (moving uniformly to the right) to another (moving uniformly to the left) at the moment it turns around. There is no single inertial frame of reference in which the second clock is stationary throughout its history, and the enormous decelerations and accelerations attended upon turning around and heading back to the first clock will be quite evident to anybody moving with the second clock.

In the frame of reference of the first clock (which uses horizontal lines of constant time) it is clear from Figure 11 that when the trip is over the second clock will have advanced only by 8 (4 on the outward journey and 4 on the inward journey) while the first clock has advanced by 10. So when the two clocks come back together the first will read 10 and the second, 8, as indicated in the figure.

Things are trickier from the point of view of the second clock, since two different inertial frames of reference are involved. In the frame moving outward with the second clock, the first clock runs slowly and advances only by 3.2 (from reading 0 to reading 3.2) during the time the second advances by 4 (from 0 to 4). This is revealed by the lower of the two tilted lines of constant time in Figure 11. Similarly, in the frame moving inward with the second clock, the first clock is also running slowly and advances only by 3.2 (from reading 6.8 to reading 10) as the second advances by 4 (from 4 to 8), as revealed by the upper tilted line of constant time in Figure 11.

The indisputable fact that the first clock reads 10 and the second reads only 8 when they are reunited makes sense from the point of view of the second clock, even though the first clock runs slowly in both the outgoing and the incoming frames. The missing 3.6 units of first-clock time ($3.6 = 10 - 2 \times 3.2$) comes from a correction that must be made in the notion of *what-the-first-clock-reads-now* when the second clock changes frames. As Figure 11 shows, at the place and time of turn-around, when the second clock reads 4, the far-away first clock *now* reads 3.2 according to the notion of simultaneity in the outgoing frame, but it *now* reads 6.8 according to the notion of simultaneity in the incoming frame. It is this adjustment, with the change of frames, of what the first clock is doing *now*, that accounts for the missing time.

(4) The essential role played by the different simultaneity conventions in different frames of reference, drops out of the story if we ask not what people moving with each clock *say* about the current reading of the other clock, but what they actualy *see* it doing. Figure 12 reproduces the clocks of Figure 11, without the lines of constant time appropriate to the three different frames of reference, but with the trajectories (dotted lines) of photons emitted by each clock as its reading changes. Since the slowing-down factor for the moving clocks is 4/5, the relative velocity of the clocks is $v = \frac{3}{5}c$, and therefore the Doppler factor,[9] $\sqrt{\frac{1+v/c}{1-v/c}}$ is 2: people watching a clock moving away from them (or moving away from the clock) at 3/5 the speed of light, will *see* it running at half its proper rate; people watching a clock moving towards them (or moving towards the clock) at 3/5 the speed of light, will *see* it running at twice its proper rate.

People with the first clock (which has the vertical line of constant position) see the light emitted by the second clock as it changes to 1, 2, 3, and 4, as the first clock reads

[9] This was discussed in Lecture Notes #7. Note that the factors of 2 and $\frac{1}{2}$ emerge automatically from the geometry of Figure 12.

2, 4, 6, and 8; they see the light emitted by the second clock as it changes to 5, 6, 7, and 8 as the first clock reads 8.5, 9, 9.5, and 10. So they see the second clock running at half its proper rate for 80% of the time and at twice its proper rate for 20% of the time. The considerable time seen running slowly overwhelms the rather brief time seen running fast, and the net effect is that the second clock has not advanced as much as the first when the journey is over.

On the other hand people with the second clock (which has the slanting lines of constant position) see the light emitted by the first clock as it changes to 1 and 2, as the second clock reads 2 and 4. They see the light emitted by the first clock as it changes to 3, 4, 5, 6, 7, 8, 9, and 10, as the second clock reads 4.5, 5, 5.5, 6, 6.5, 7. 7.5, and 8. So they see it running at half its proper rate for half the time and at twice its proper rate for half the time. Since it is seen running at twice its proper rate for half the time, this already insures that it will have advanced by as much as the second clock when they are back together, and because it is seen running at half its proper rate for the remaining half, it will have advanced by 25% more than that when they are back together.

(5) Alice runs toward the front door of a long narrow barn, that stretches away from her. She carries a long horizontal pole that points towards the door. The proper length of the pole is greater than the proper length of the barn, so if the pole were stationary it could not fit in the barn. But Alice runs so fast that in the barn frame the shrinking of the pole makes it shorter than the barn, and the pole fits comfortably into the barn. In Alice's frame, on the other hand, the pole retains its proper length and it is the barn, moving toward Alice, that suffers a length contraction, so it is even more impossible (if something impossible can become more impossible) for the pole to fit in the barn.

What's going on here? To avoid having to worry about the pole crashing into the back of the barn, or the considerable complications introduced by the difficult process of slowing down and stopping inside the barn a pole that moves at a speed comparable to the speed of light, let's suppose that the barn has a rear door as well as a front door, so the pole can continue moving uniformly out of the barn, without ever changing its speed. Is there or is there not a time when the pole is in the barn?

Notice the appearance of the crucial word *time*. The resolution of this apparent paradox is that whether or not a moving pole fits in a barn does indeed depend on your frame of reference. For "the pole is in the barn" really means "all the parts of the pole are between the front and rear door of the barn at the same time". Since different parts along the pole are in different places, and since different frames of reference use different conventions in determining whether events in different places are simultaneous, there can indeed be a legitimate disagreement about whether there are moments of time at which all the different parts of the pole are between the two doors of the barn.

The space-time diagram in Figure 13 makes it clear what is really going on. Suppose that each barn door is closed except when the pole is actually in the doorway. The two vertical lines represent these doors, which are shut when the lines are solid and open when they are dotted. The points making up the moving pole lie in the grey shaded region, which is bounded on the left by the space-time trajectory of the left end of the pole and on the right by the space-time trajectory of the right end.

Lines of constant time in the barn frame are horizontal. Two such lines are shown. The lower horizontal line contains the moment when the rear end of the pole has entered the barn, and demonstrates that at that moment the front end of the pole has yet to leave the barn. The upper horizontal line contains the moment when the front end of the pole reaches the rear door of the barn, and demonstrates that at that moment the rear end of the pole is well within the barn. At all barn-frame times between those associated with the two horizontal lines, the pole is inside the barn, and both barn doors are shut.

But lines of constant time in the pole frame tilt upward to the right. Two such lines are shown. The lower line contains the moment when the front end of the pole reaches the rear of the barn, and demonstrates that at that moment the rear end of the pole has yet to enter the barn. The upper line of constant barn-frame time contains the moment when the rear end of the pole finally enters the barn. At that moment of barn-frame time the front end has already left the barn. There is no moment of pole-frame time during which both barn doors are shut. In the pole frame, the pole is never entirely within the barn.

The important lesson of the pole-in-barn paradox is that even so innocent sounding a sentence as "The pole is shut up in the barn" involves an implicit judgment about the simultaneity of events in different places when it is applied to a moving pole.

Appendix: Scale factors and the invariance of the interval.

We first establish the connection between the scales Alice and Bob use on their lines of constant position (or lines of constant time). A segment of a line of constant Alice position separating events a time T apart in her frame, is related to a segment of a line of constant Bob position separating events a time T apart in his, by the following rule (illustrated in Figure 13 below): *The rectangles of photon trajectories having the segments for diagonals have the same area.*

This rule is illustrated in Figure 14. Part (a) of Figure 13 shows two the two moments at which a clock, stationary in Alice's frame, reads 0 and T. The two moments in the history of the clock lie on a line of constant Alice position a distance $\mu_A T$ apart. The two photon trajectories emerging from the lower picture of the clock and the two entering the upper picture of the clock form a rectangle which has as its diagonal the segment of Alice's line of constant time connecting the clocks. Part (b) of Figure 14 shows the same state of affairs for a clock stationary in Bob's frame. The length $\mu_B T$ of the segment of Bob's line of constant position connecting the two moments in the history of his clock exceeds the length $\mu_A T$ of the corresponding segment associated with Alice's line of constant position. But it can be shown that the areas of the two surrounding dashed rectangles are exactly the same.

To see why, take the case in which the two clocks are in the same place when they read 0. This is illustrated in Figure 15.[10] Let an observer moving with Alice's clock *look at* Bob's clock at the moment Alice's reads T, and let an observer moving with Bob's clock *look at* Alice's at the moment Bob's reads T. Each will see the other's clock reading *the same* earlier time t.[11] A glance at the figure reveals that the ratio h/H of the short side of Bob's rectangle to the short side of Alice's, is the same as the ratio[12] t/T as determined from Alice's line of constant position, while the ratio b/B of the long side of Alice's rectangle to the long side of Bob's, is the same as the ratio[13] t/T as determined from Bob's line of constant position. But if $h/H = b/B$ then

$$hB = bH. \tag{10.2}$$

The left side of (10.2) is the area of Bob's rectangle (look at the figure!) and the right side is the area of Alice's. This is what we wished to establish.

That the equality of the rectangles leads immediately to the equality of the product $\lambda\mu$ of scale factors follows from the fact that four copies of either of the two identical triangles making up either rectangle (Part (a) of Figure 16) can be reassembled into a rhombus whose sides have length μT and are a distance λT apart (Part (b) of Figure 16). The area of the rhombus is $\lambda\mu T^2$, so the area of the rectangle is $\frac{1}{2}\lambda\mu T^2$ (where one uses λ_A and μ_A for Alice's rectangle and λ_B and μ_B for Bob's. Since Alice's and Bob's rectangles have the same area, this establishes (1) that the product $\lambda\mu$ is independent of frame of reference and (2) that the relation between the area A of either rectangle in Figure 14 and the time T between the clock present at the two events on its opposite corners is just

$$T^2 = A/(\tfrac{1}{2}\lambda\mu). \tag{10.3}$$

Abstracting from this, we conclude that the area of a rectangle of photon trajectories with two time-like separated events at opposite vertices, is just a frame-independent scale factor $(\frac{1}{2}\lambda\mu)$ times the square of the time between the two events in the frame in which they happen at the same place.[14] This is precisely the *squared interval* I^2 between the

[10] Figure 15 results from simply sliding (without rotating) part (b) of Figure 14 over to part (a), to bring the two clocks reading 0 into coincidence, and then adding a few things.

[11] Each will see the same time, because each looks after the same time (T) has passed on his or her own clock, and each regards the other's clock as moving away at the same speed — i.e. the relation between Alice, Bob, and their clocks is completely symmetric.

[12] Actually, the ratio $\mu_A t/\mu_A T$, but the common scale factor μ_A has no effect on the ratio.

[13] Actually $\mu_B t/\mu_B T$.

[14] Because of the explicit symmetry of the diagrams under the interchange of space and time we can also conclude that the area of the rectangle of photon trajectories with two space-like separated events at opposite vertices, is that same invariant scale factor times the square of the distance between the two events in the frame in which they happen at the same time.

events:

$$I^2 = A/(\tfrac{1}{2}\lambda\mu). \tag{10.4}$$

Finally one can see directly from Figure 17 and its caption that the squared interval between two time-like separated events is indeed the difference of the square of the time between them and the square of the distance between them, regardless of the frame in which that time and distance are evaluated.

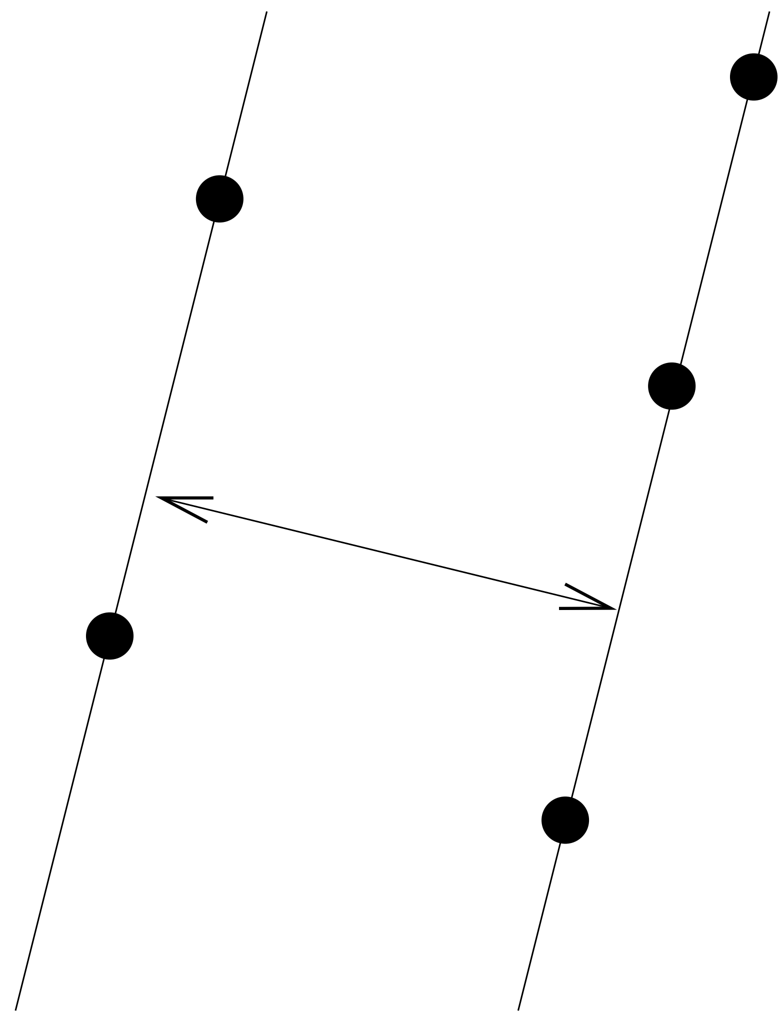

Figure 1. Two lines of constant position in Alice's frame. The two black dots on the line on the left represent two events that happen in a single place (but at different times) according to Alice; the three black dots on the line on the right represent three other events that happen in a single place, different from the place of the two events on the left. The distance between two such lines in the diagram (indicated by the double-headed arrow) is proportional to the actual distance in Alice's frame between the two places they represent. Such a diagram is characterized by a scale factor λ which specifies, for example, the number of centimeters on the page between lines representing places a foot apart in space.

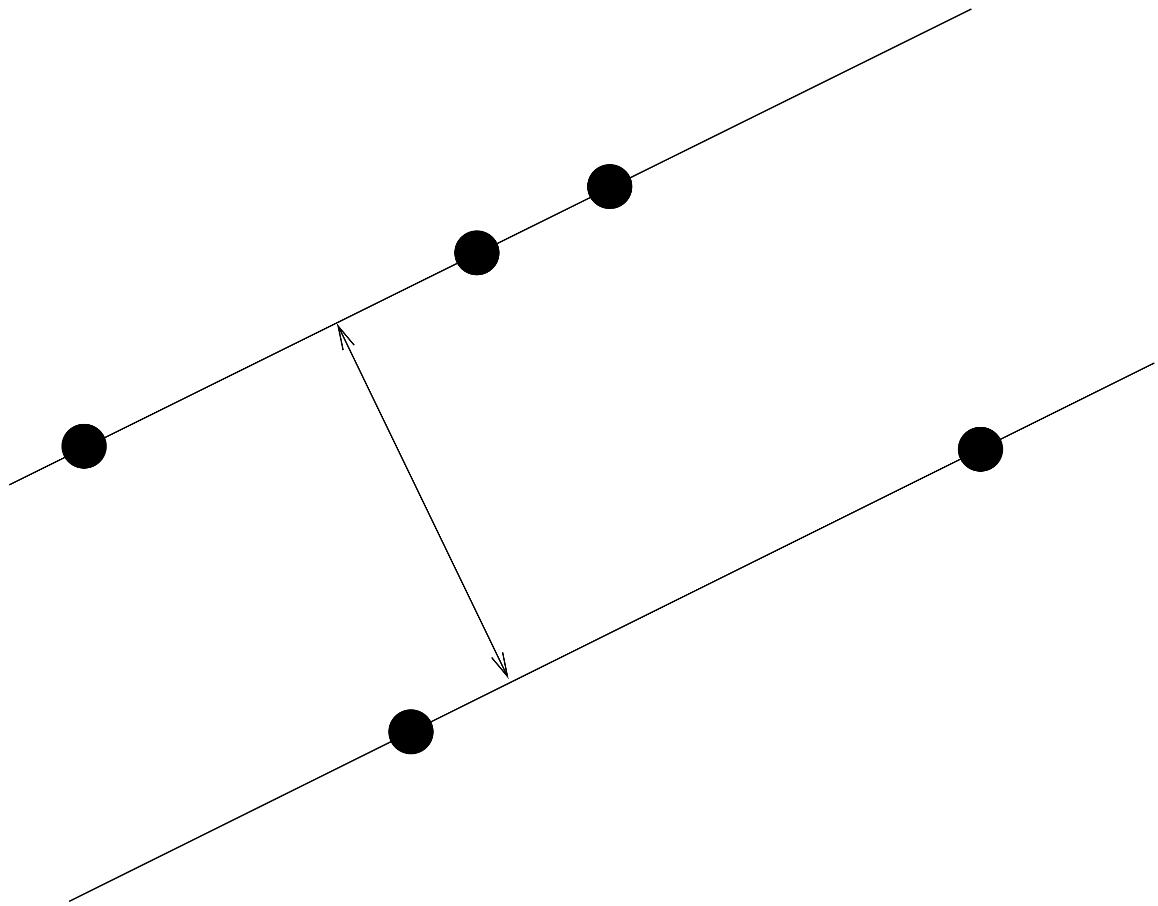

Figure 2. Two lines of constant time in Alice's frame. The two black dots on the lower line represent two events that happen in at a single time (but in different places) according to Alice; the three black dots on the upper line represent three other events that happen at a single time, different from the time of the two on the lower line. The distance between two such lines in the diagram (indicated by the double-headed arrow) is proportional to the actual time in Alice's frame between the two moments of time they represent. Such a diagram is characterized by a scale factor λ which specifies, for example, the number of centimeters on the page between lines representing events a nanosecond apart in time.

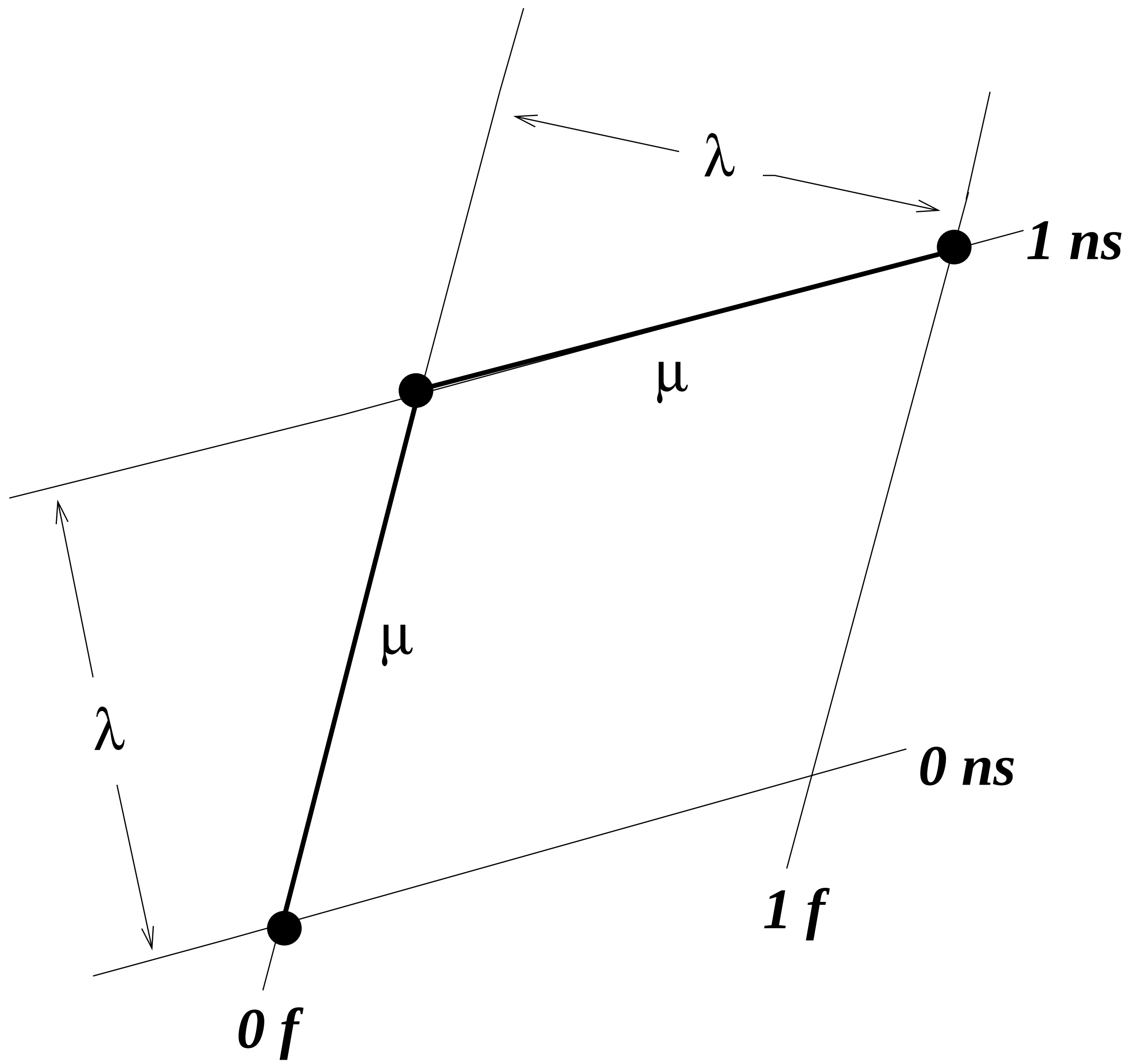

Figure 3. The scale factors λ and μ. The parallel lines tilting slightly upward to the right are lines of constant time; events represented by points on the upper line happen one nanosecond after events represented by points on the lower. The parallel lines tilting steeply upward to the right are lines of constant position; events represented by points on one line happen one foot away from events represented by points on the other. The scale factor λ is the distance in the diagram between the lines of constant position or between the lines of constant time. The scale factor μ is the length in the diagram of the (more heavily drawn) segments of the lines of constant time and position between the events represented by the black dots.

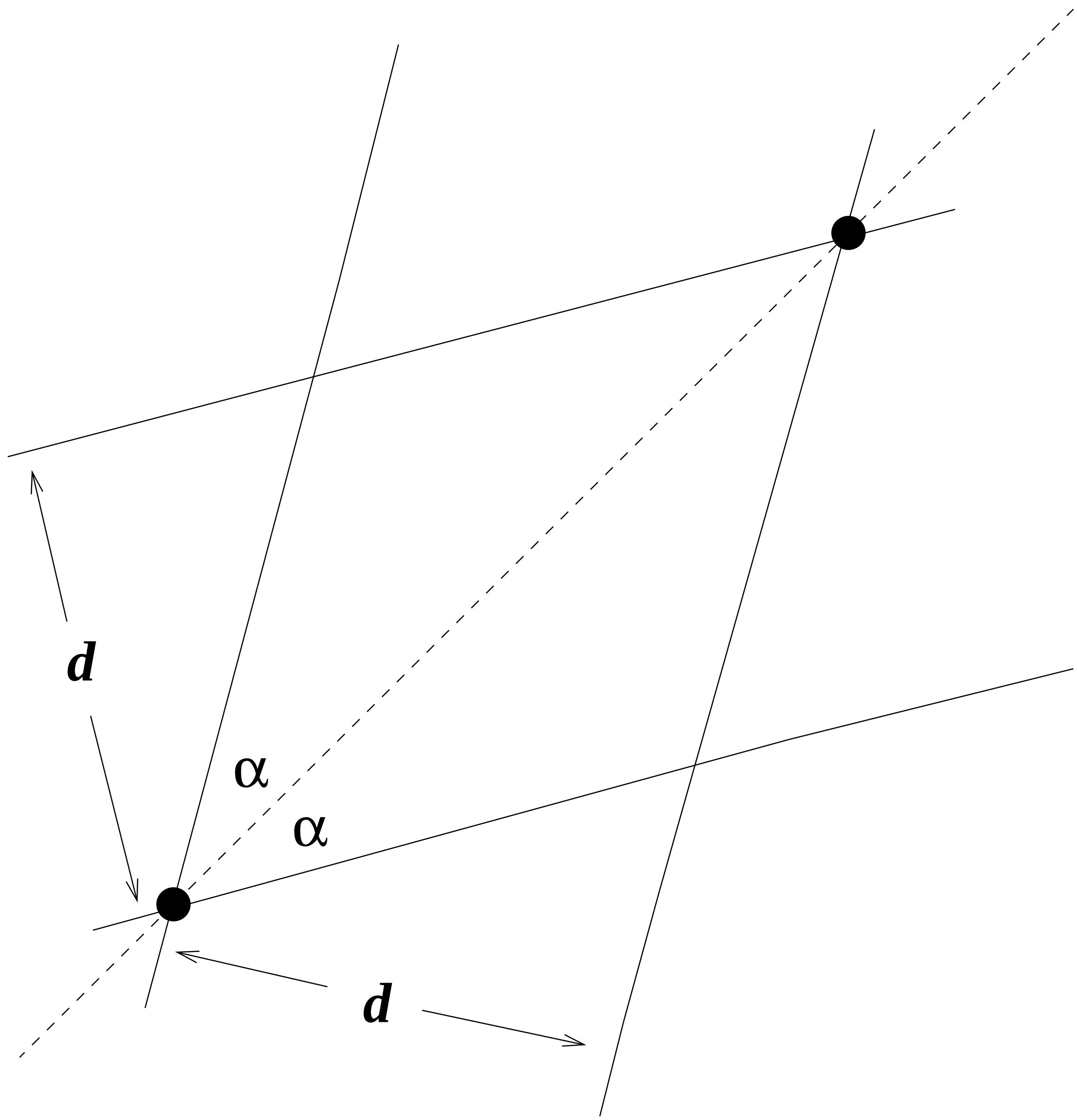

Figure 4. The dashed line represents the space–time trajectory of a photon. The black dots represent two events in the history of that photon. A line of constant position slants steeply upward through each dot, and a line of constant time slants slightly upward through each. Because the photon moves one foot every nanosecond, the distance d between the lines of constant position is the same as the distance d between the lines of constant time. The parallelogram formed by the four lines is therefore a rhombus (i.e. all four sides have the same length), the dashed photon line is the diagonal of that rhombus, and the symmetry of the figure requires the angles labeled α to be the same.

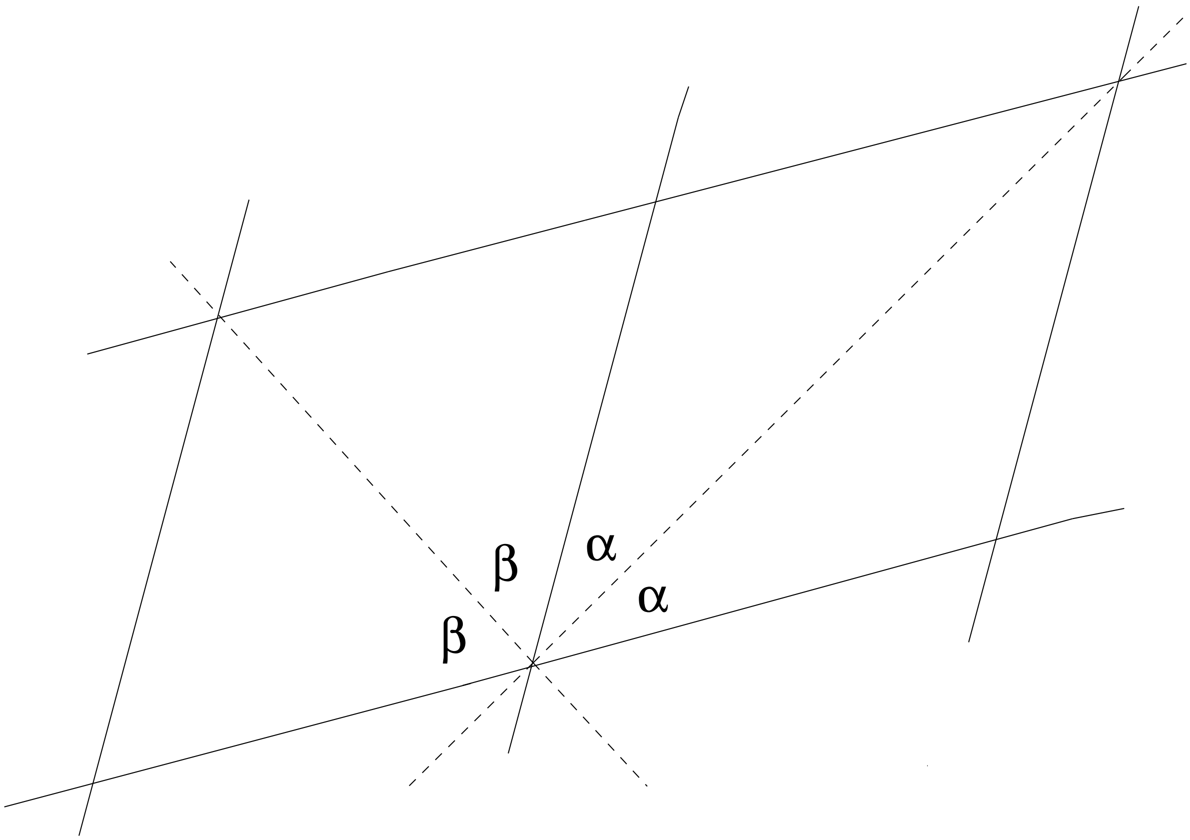

Figure 5. Figure 4 is redrawn (without the black dots) and extended to show the space–time trajectory of a second photon travelling in the opposite direction. Because the new dashed line is also a photon trajectory it also bisects the angle between the lines of constant time and position. Since $2\alpha + 2\beta = 180^\circ$, the angle $\alpha + \beta$ between the two photon lines is 90°.

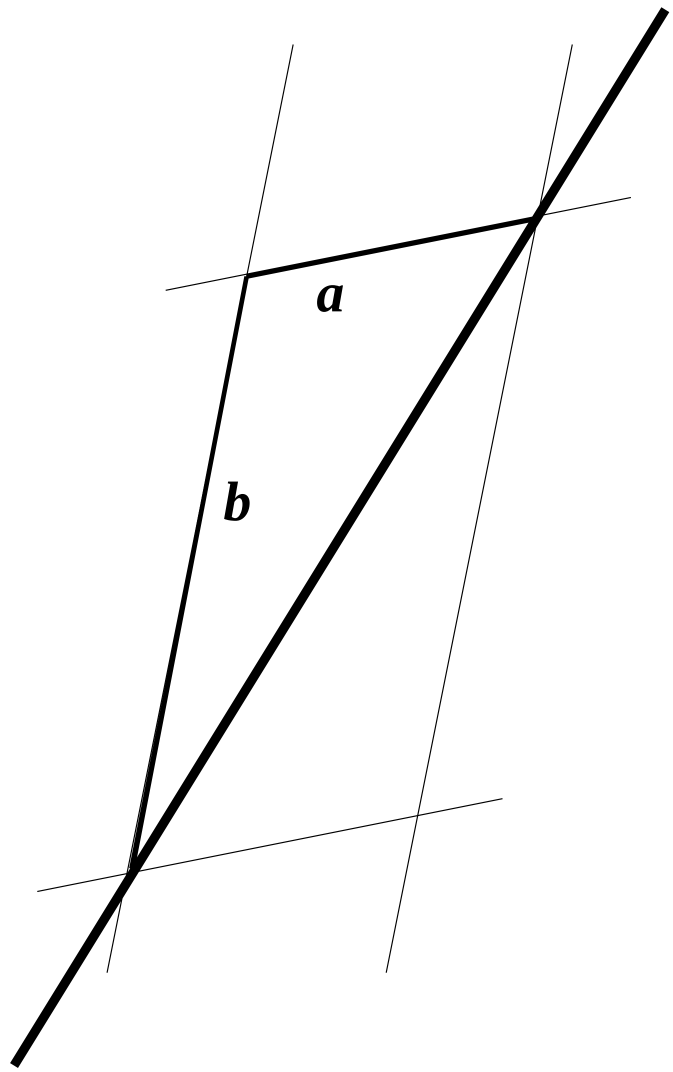

Figure 6. The very heavy line is the space-time trajectory of an object stationary in Bob's frame of reference — i.e. a line of constant position, according to Bob. The lighter lines are lines of constant Alice-time and Alice-position drawn through two events on the heavy trajectory. The lengths in the diagram of the darkened segments of those lines are a and b. The velocity of Bob with respect to Alice is $v = a/b$, since in Alice's frame the position of the object changes by a distance μa in a time μb. Note that a/b is also the ratio of the distance between the lines of constant Alice–position to the distance between the lines of constant Alice–time.

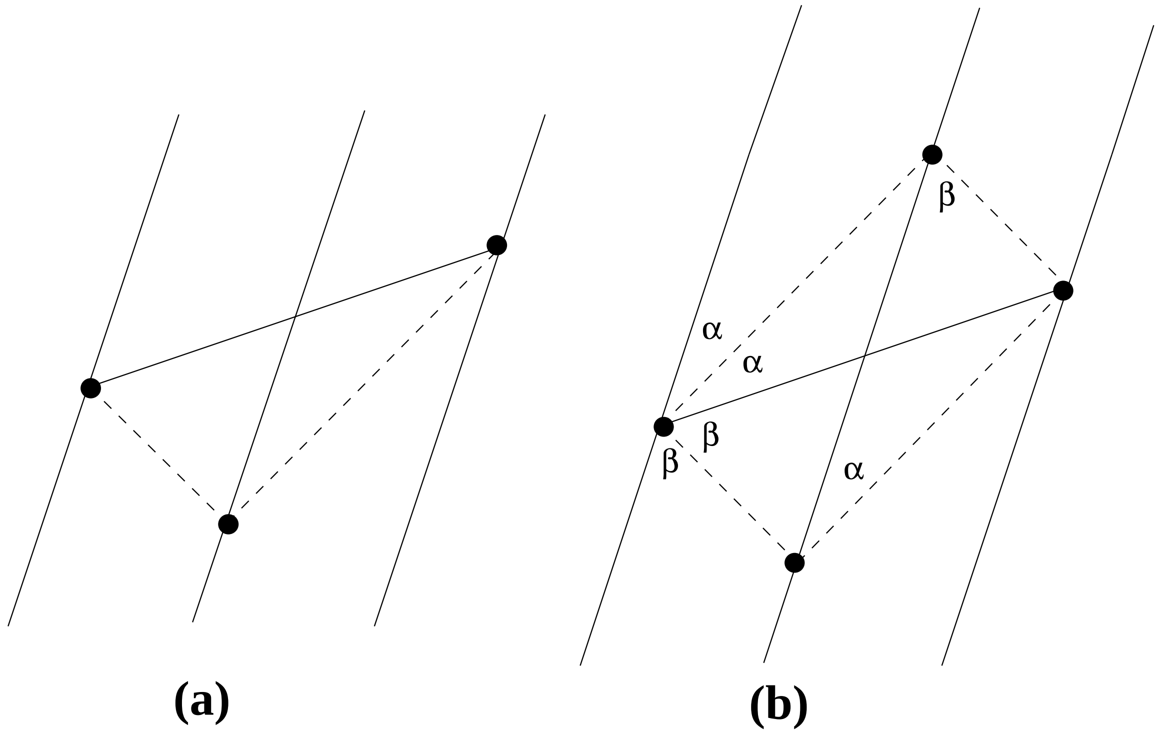

Figure 7. The diagram is drawn by Alice. (a) The three equally spaced parallel lines of constant position are the two ends and the middle of a train that is stationary in Bob's frame of reference. They establish the direction Bob's lines of constant position must have in Alice's diagram. The lowest black dot represents the production of two oppositely directed photons at the middle of the train. The dashed lines are the space-time trajectories of the photons. The other two black dots represent the arrival of each photon at an end of the train. Since both photons move at the same speed in Bob's frame of reference and since the train is stationary in Bob's frame, the photons arrive at the ends of the train at the same Bob–time — i.e. the line joining the upper two dots is a line of constant time in Bob's frame. (b) If the photons are reflected back toward the center of the train when they reach the two ends, they will arrive there at the same time in the event represented by the highest black dot, the four photon lines forming a rectangle. It is evident from the symmetry of the rectangle that the two angles inside the rectangle labled α are equal, as are the two angles inside the rectangle labeled β. Since the two labeled angles outside the rectangle are just spatial translations of two correspondingly labeled angles within it, it follows that both of the photon trajectories passing through the left–most black dot bisect the lines of constant Bob-time and Bob-position passing through that dot.

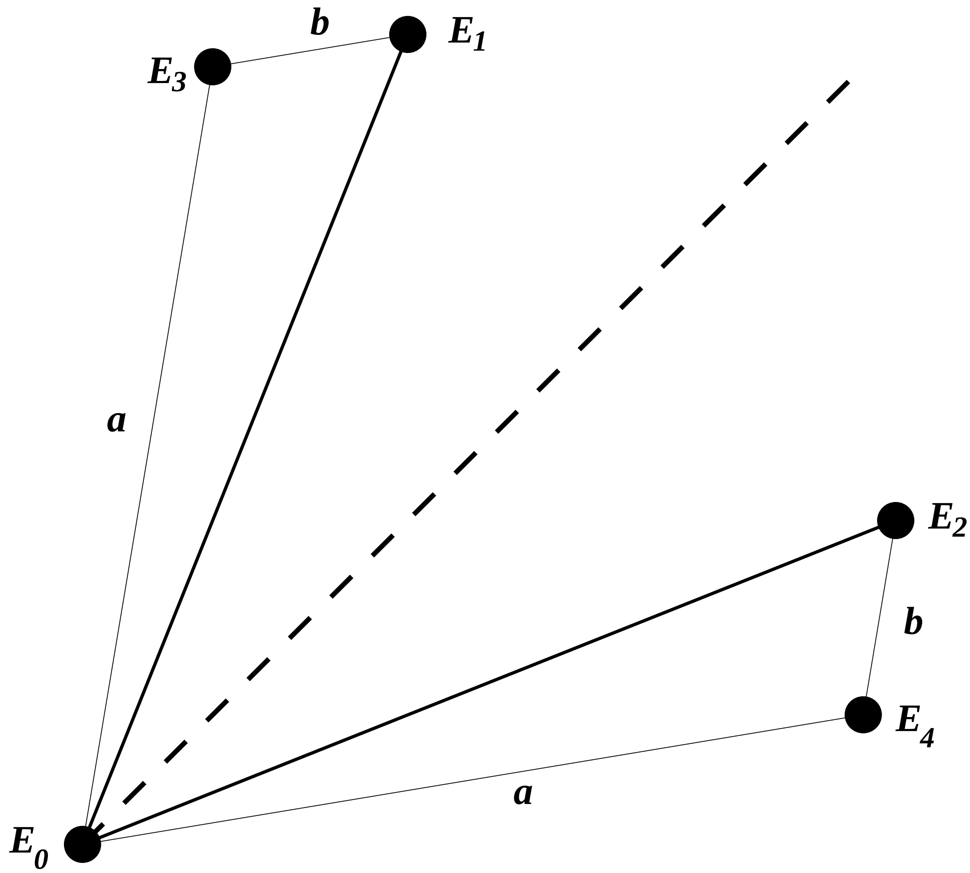

Figure 7a. The four light solid lines are lines of constant position and constant time in Alice's frame. The two heavy solid lines are lines of constant position and constant time in Bob's frame. Bob's frame moves with speed v with respect to Alice's. Consider first the part of the diagram above the heavy dashed 45° photon line. The events E_0 and E_1 lie on a line of constant Bob-position and can therefore be two events in the history of something moving with speed v, according to Alice. Since E_3 lies on the same line of constant Alice-time as E_1, according to Alice the time T between E_1 and E_0 is the same as the time μa between E_3 and E_0. Since E_3 lies on the same line of constant Alice-position as E_0, the distance D between E_1 and E_0 is the same as the distance μb between E_1 and E_3. We therefore have $v = D/T = b/a$. The part of the diagram below the photon line is just the mirror image in that line of the part above it, so the ratio of b to a in the lower half continues to be v. But in the lower half of the diagram the ratio of b to a is the ratio of Alice's time T to her distance D between the two events E_0 and E_2 lying on Bob's line of constant time. (For μb is Alice's time between E_2 and E_4 which is the same as the time between E_2 and E_0, and μa is Alice's distance between E_4 and E_0 which is the same as the distance between E_2 and E_0.) So although Bob says E_0 and E_2 happen at the same time (since they lie on the same line of constant Bob-time) Alice says the time and distance between them are related by $T/D = b/a = v$: the time between them in nanoseconds is v (in feet per nanosecond) times the distance between them in feet.

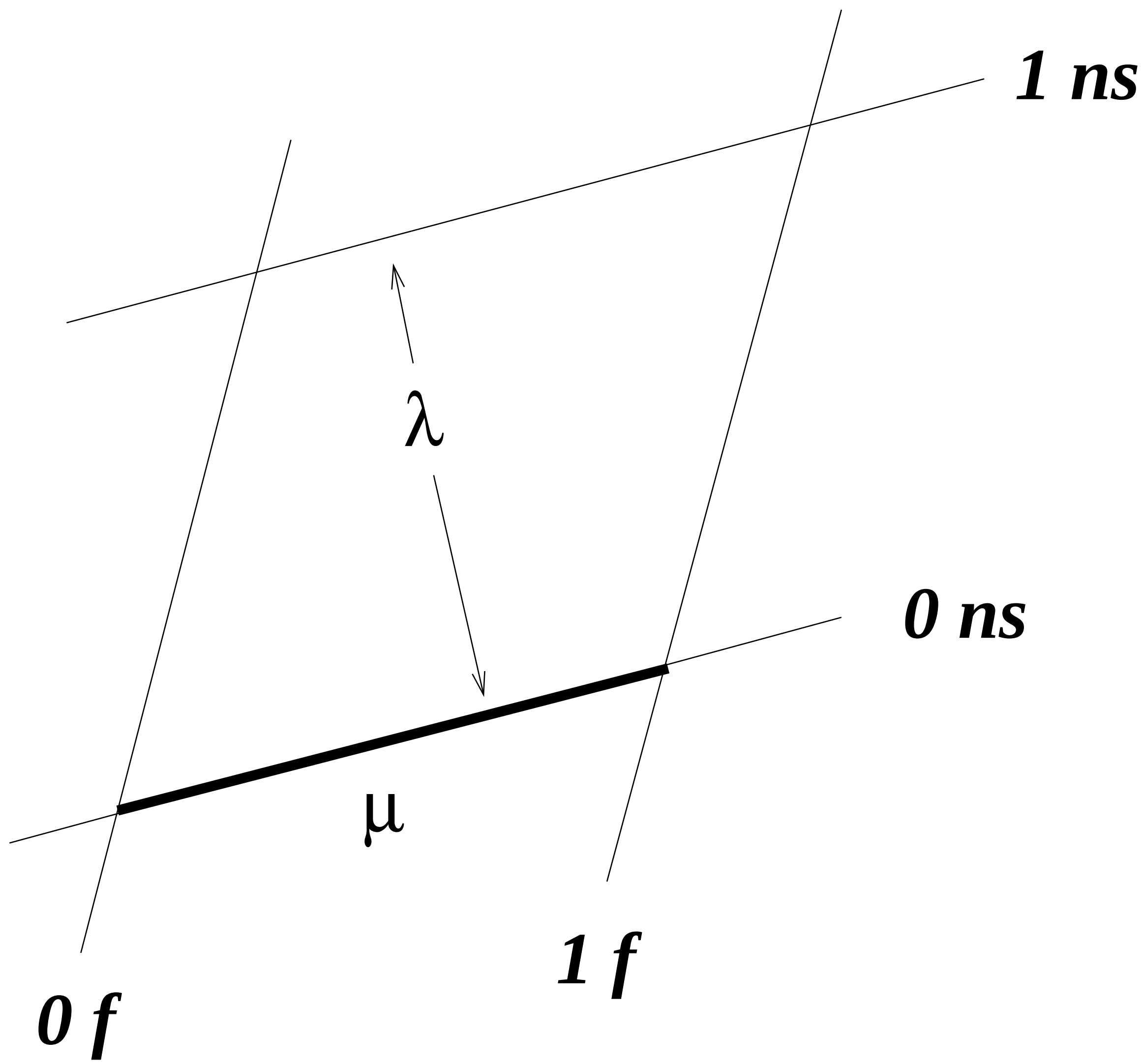

Figure 8. The unit rhombus for some frame of reference. The lines labeled 0 ns and 1 ns represent events one nanosecond apart and the lines labeled 0 f and 1 f represent events one foot apart. Because the distance in the diagram between the two lines of constant time — regarded as the height of the rhombus — is the scale factor λ and the heavier portion of the lower line of constant position — regarded as the base of the rhombus — is the scale factor μ, the area of the rhombus — its base times its height — is just the product $\lambda\mu$.

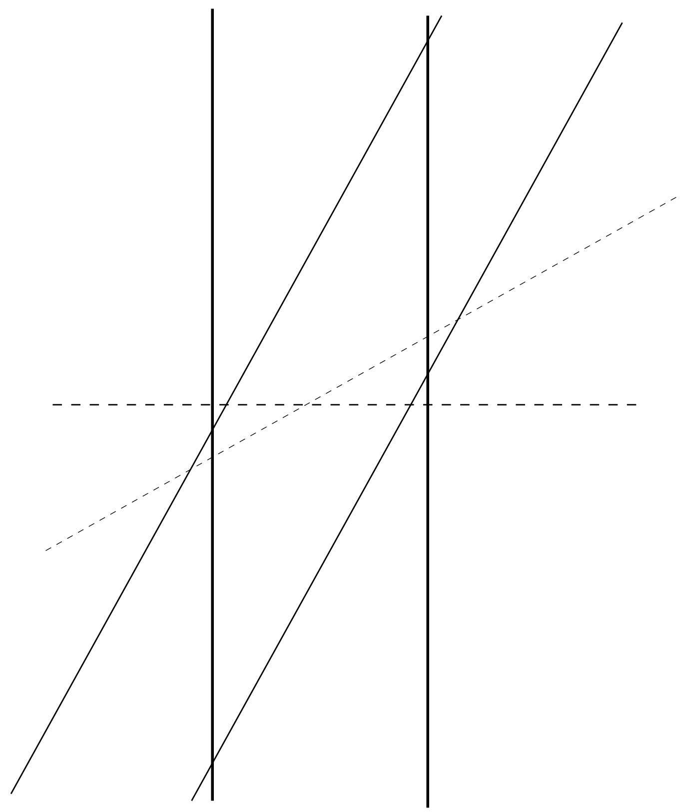

Figure 9. The two vertical solid lines are the left and right ends of a stick. The two solid lines that tilt upward to the right are the left and right ends of a second stick that moves to the right past the first stick. The horizontal dashed line is a line of constant time in the frame in which the first stick is stationary. Note that both ends of the first stick extend beyond both ends of the second along that line of constant time, thereby establishing that the first stick is longer than the second in its proper frame. The dashed line that tilts upward to the right is a line of constant time in the frame in which the second stick is stationary. (Note that it tilts away from the horizontal by the same amount that the lines representing the ends of the second stick tilt away from the vertical.) Along this tilted line of constant time both ends of the second stick extend beyond both ends of the first stick, thereby establishing that the second stick is longer than the first in its proper frame.

The figure vividly demonstrates that what one means by *a stick at a given moment of time* depends on the frame of reference in which the stick is described, and that it is this that makes it possible for people using the proper frame of either stick to maintain that the other stick is shorter.

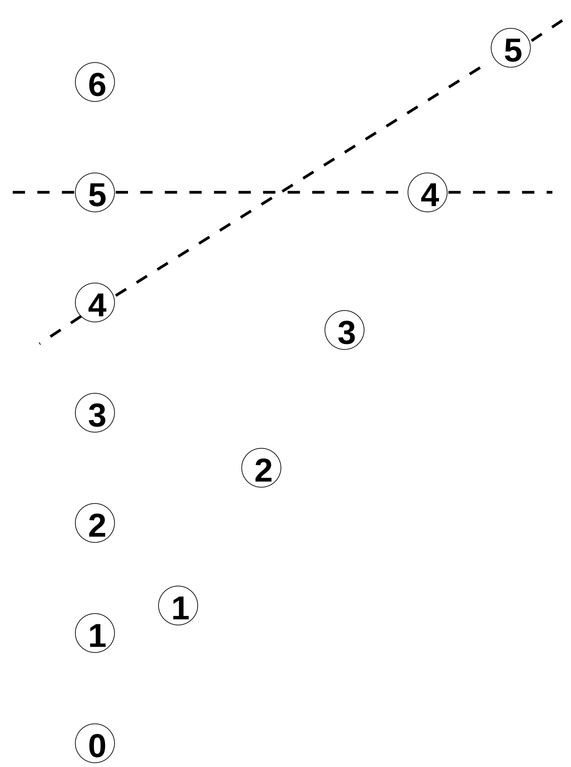

Figure 10. Several moments in the histories of two uniformly moving clocks (represented by circles with numbers inside giving their readings.) Both clocks read 0 at the same place and time and are represented by just a single circle in the figure. Subsequent readings of the first clock (1-6) are shown on the set of circles uniformly spaced along a vertical line; subsequent readings of the second clock (1-5) are shown on the set of circles that lie on a line sloping upward to the right. The horizontal dashed line is a line of constant time in the frame of the first clock. In that frame the second clock has advanced from 0 to 4 in the time it took the first to advance from 0 to 5, so the second clock is running slowly by a factor $s = 4/5$. The slanting dashed line is a line of constant time in the frame of the second clock (and tilts away from the horizontal by the same amount that the line along which the pictures of the second clock lie tilts away from the vertical.) In that frame the first clock has advanced from 0 to 4 in the time it took the second to advance from 0 to 5.

The figure makes clear that how one compares the rates of two clocks in relative motion depends on how one judges whether two events in different places are simultaneous. This is what makes it possible for people using the proper frame of either clock to maintain that the other clock is running slowly.

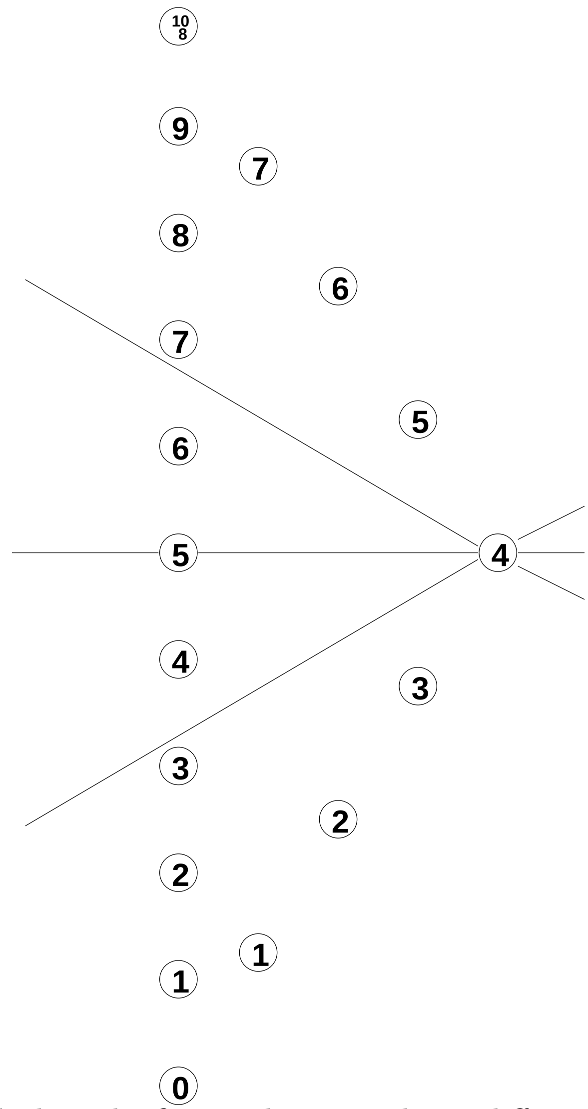

Figure 11. Two identical clocks. The first is shown at eleven different moments along the vertical line, as its reading advances from 0 to 10. The second moves away from the first as it advances from 0 to 4; it then moves back to the first, as its reading advances from 4 to 8. At the bottom and top of the figure both clocks are at the same place at the same time and are represented by a single circle. The first clock is stationary in a single inertial frame of reference. Since lines of constant position are vertical in that frame, lines of constant time are horizontal. Consequently it is evident from the figure that in the proper frame of the first clock, the outward and inward journeys of the second clock each take 5 seconds, during each of which the second clock only advances by 4 seconds.

Two other lines of constant time are shown passing through the point at which the second clock begins its return journey. One line (going downward to the left) is appropriate to the proper frame of the second clock during its outward journey; the other (going upward to the left) is appropriate to the proper frame of the second clock during its inward journey. Note that just before the second clock changes frames, the time on earth in the outward-going frame is about 3.2 seconds. Just after the second clock has changed frames, the time on earth in the inward-going frame is about 6.8 seconds.

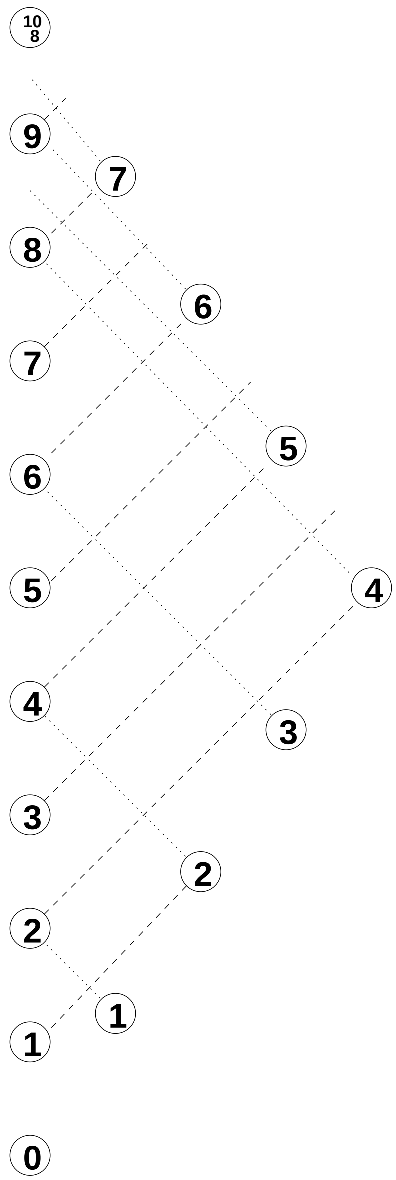

Figure 12. This repeats figure 9, but without the lines of constant time, and with many photon trajectories indicating what somebody moving with either clock *sees* the other clock doing. Each clock emits a flash of light each time its reading changes, and those flashes are seen by people moving with the other clock. People at the position of the first clock (vertical line of constant position) see the second clock advancing at *half* its normal rate during the 8 seconds they are watching the flashes emitted on the outward journey, and they see the second clock advancing at *twice* its normal rate during the 2 seconds they are watching flashes emitted by the second clock on its inward journey.

People moving with the second clock, on the other hand, see the first clock running at *half* its normal rate during the four seconds of their outward journey and at *twice* its normal rate during the four seconds of the inward journey.

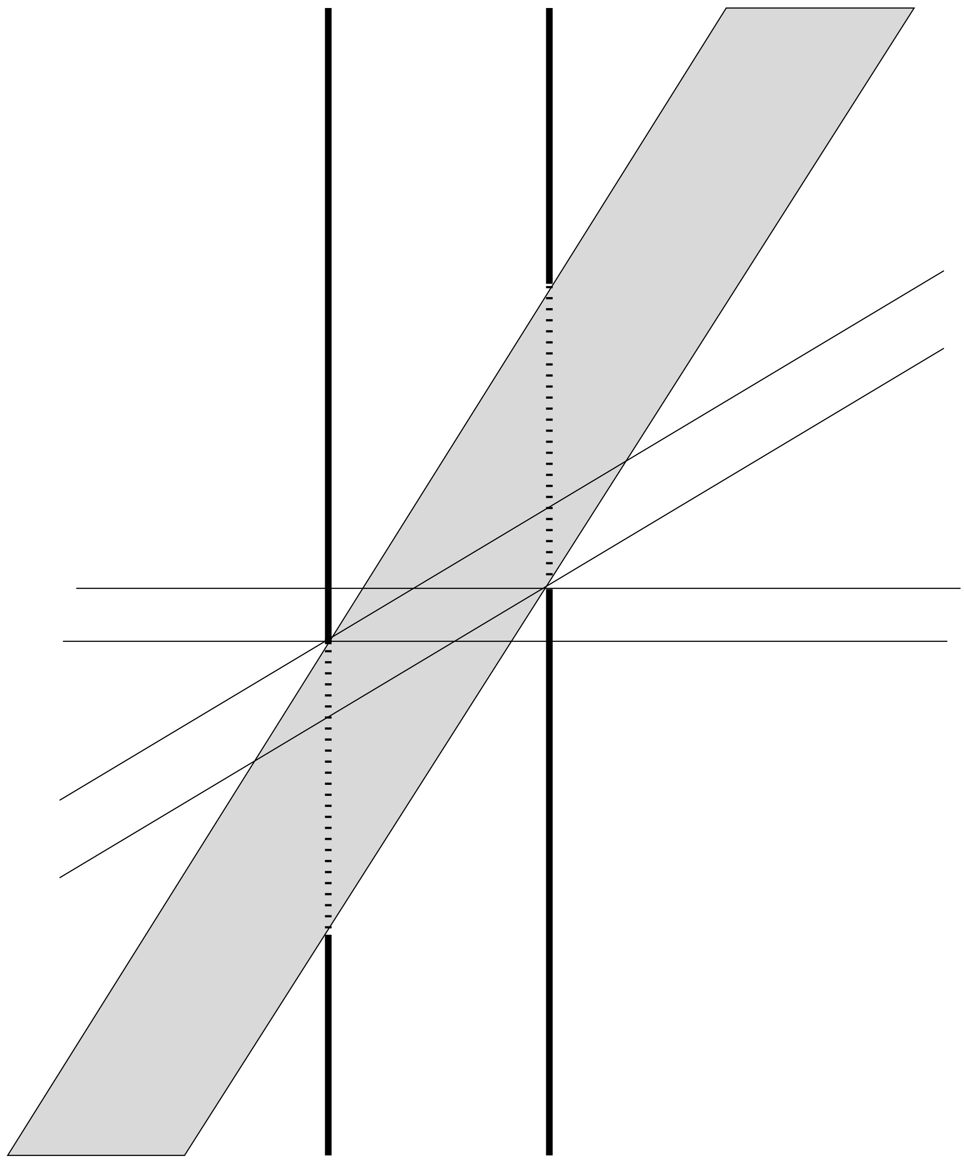

Figure 13. The pole-in-barn paradox. The heavy vertical lines are the space-time trajectories of the left and right doors of the barn, which are shut when the lines are solid and open when the lines are dashed. The lines bounding the shaded grey region are the space-time trajectories of the left and right ends of the pole. The grey region itself represents the interior points of the pole. The two horizontal lines are lines of constant time in the barn frame. They demonstrate that there is a range of barn-frame times when the pole is entirely inside the barn and both doors are shut. The two lines slanting up to the right (but less steeply than the lines representing the ends of the pole) are lines of constant time in the pole frame. They demonstrate that there is a range of pole-frame times when the pole extends all the way through the barn and through both (open) barn doors.

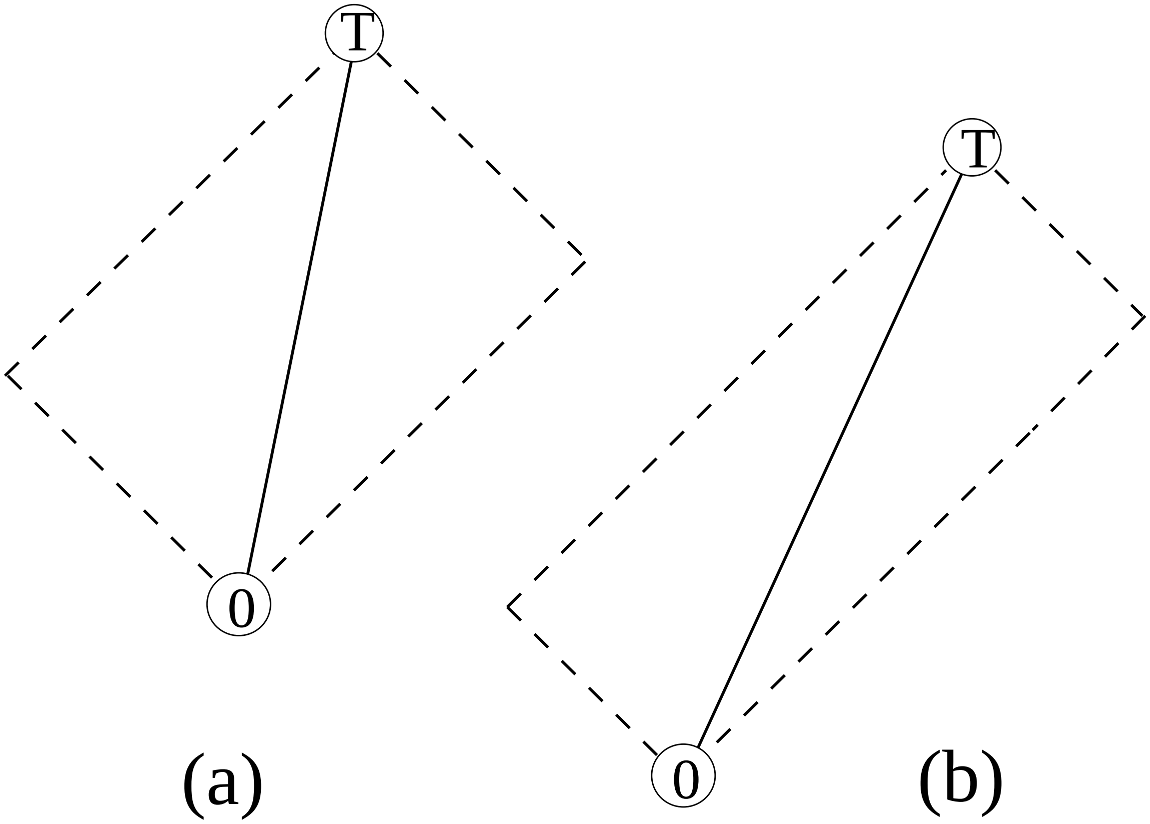

Figure 14. (a) A line of constant position in Alice's frame of reference, separating two events that are a time T apart in Alice's frame. The line can be viewed as the space-time trajectory of a clock that is stationary in Alice's frame, reading 0 at the first event and T at the second. (b) The same as (a), but for a different pair of events and different clock that is stationary in Bob's frame of reference. Note that the line connecting the events in which the clock stationary in Bob's frame reads 0 and T (which has length $\mu_B T$) is longer than the corresponding line in Alice's frame (which has length $\mu_A T$ — i.e. Alice and Bob use different scale factors μ to relate relate separation in time to distance along lines of constant position. However the *areas* of the two rectangles formed by photon trajectories emerging from the events are the same. This is established in Figure 15.

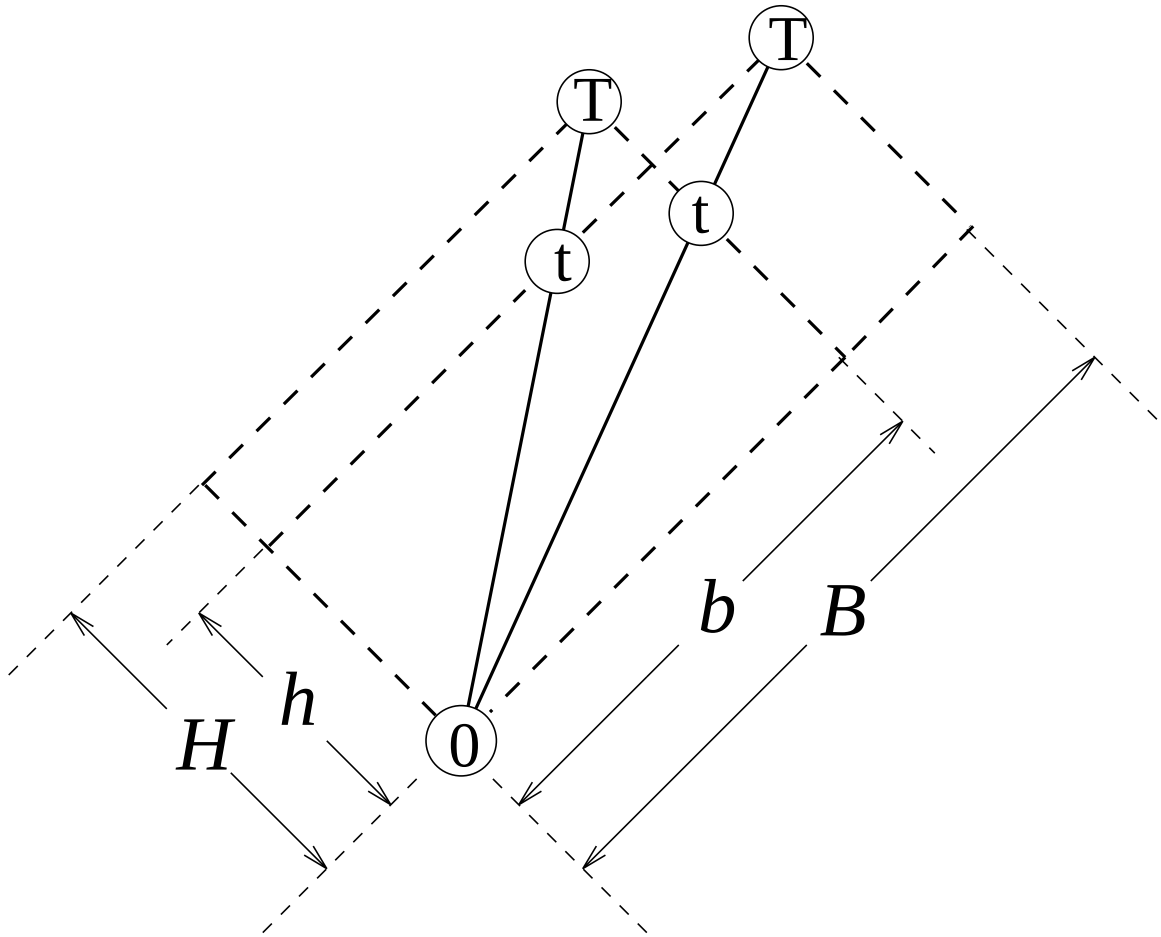

Figure 15. The two parts of figure 14 have been slid together (without rotating either) so that the two clocks reading 0 happen at the same place and at the same time. At the moment each clock reads T, somebody with the clock looks at the other clock and sees it reading t. The ratio of t and T along either line of constant position is just the ratio of the distances in the diagram, μt and μT, from those moments in the history of the clock back to the moment at which the clock reads 0. (One uses μ_A for Alice's line and μ_B for Bob's.) It is evident from the figure that this ratio is also the same as the ratio of h to H or the ratio of b to B. But if $h/H = b/B$ then $Bh = bH$ — i.e. the two rectangles transported from Figure 14 to Figure 15 have the same area.

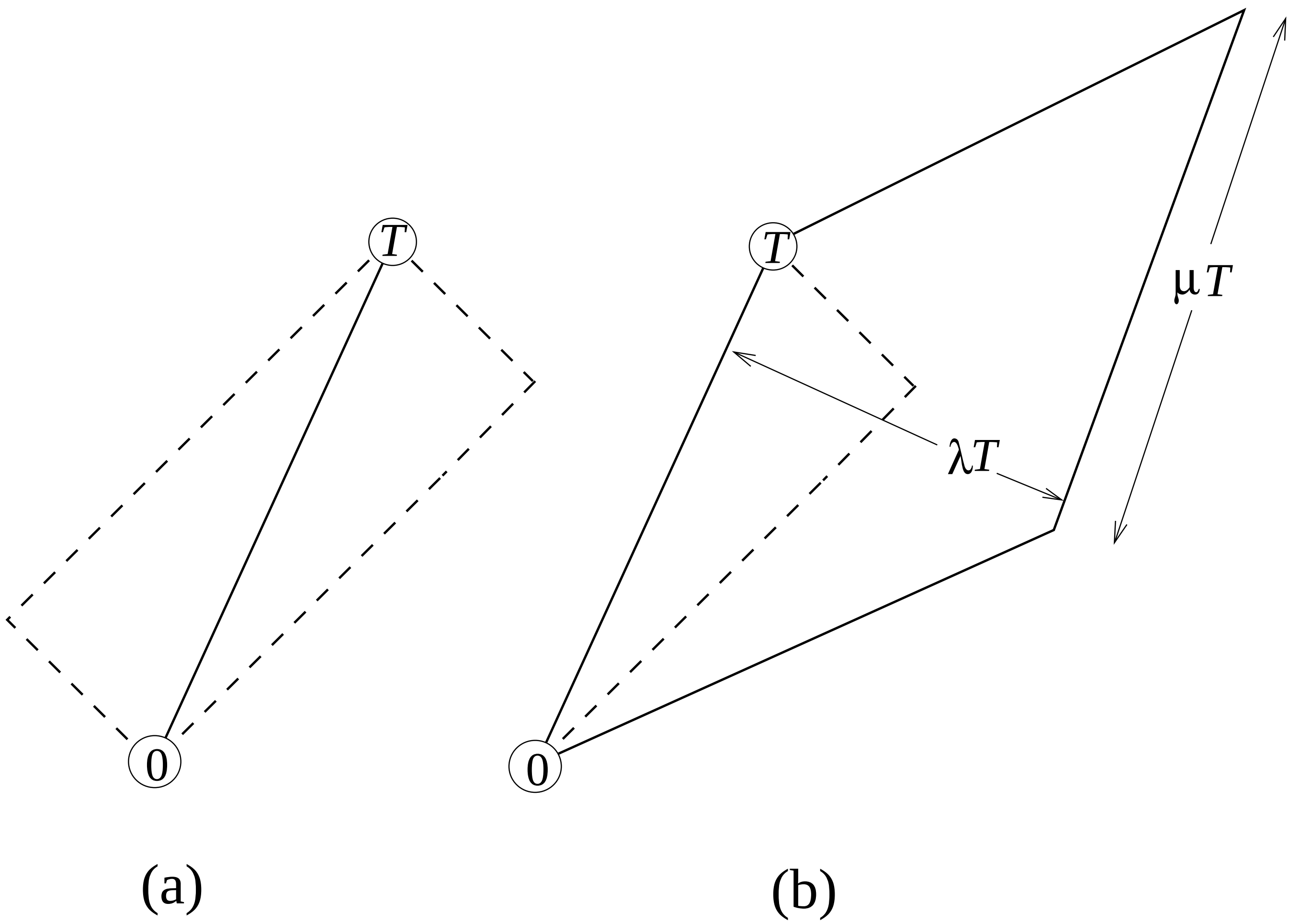

Figure 16. The area of either of the photon rectangles in Figure 14, shown here in part (a), is half the area of the rhombus shown in in part (b). (For the rhombus can be assembled out of four of the right triangles, two of which make up the rectangle.) But the area of the rhombus in part (b) is the length μT of a side times the distance λT between sides. So the area of the rectangle in part (a) is $\frac{1}{2}\lambda\mu T^2$.

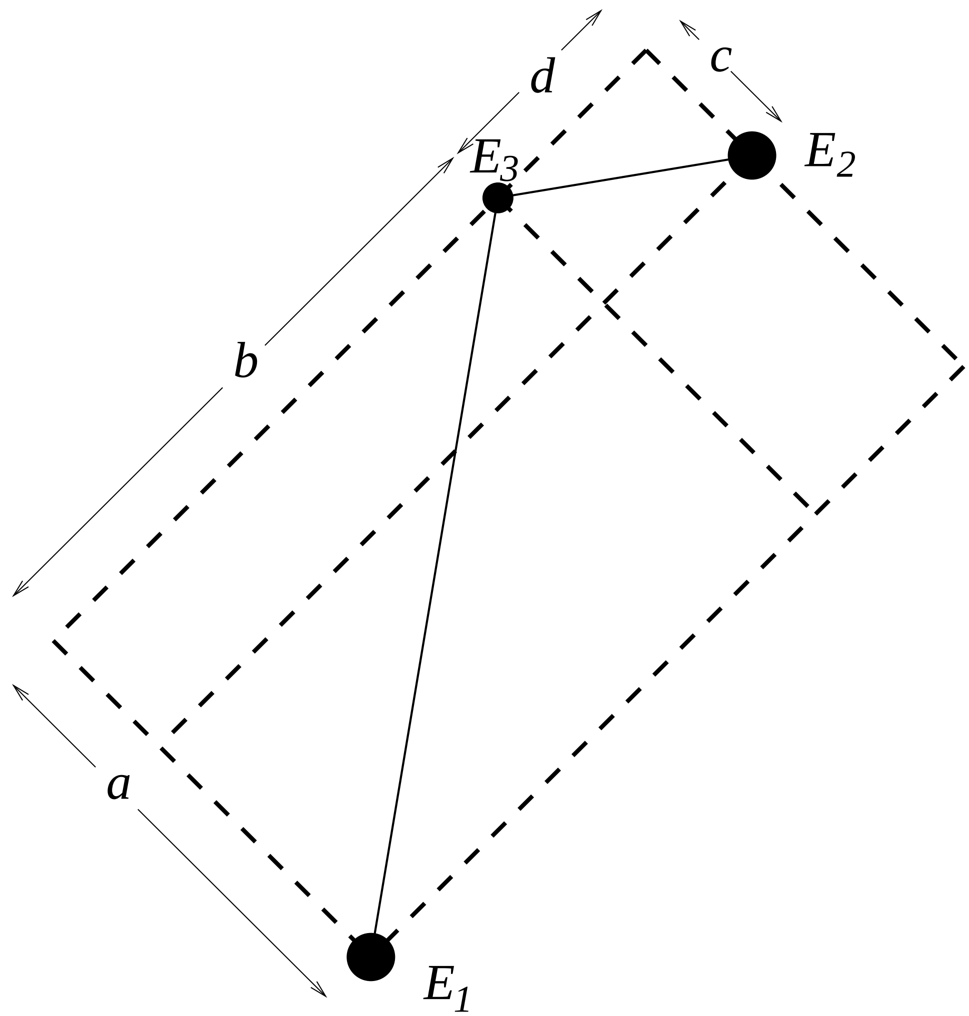

Figure 17. The two large black dots are two events E_1 and E_2. The dashed lines are photon lines. The two solid lines are lines of constant time and constant position in Alice's frame of reference, and therefore the right triangle with sides d and c is just a scaled down version of the right triangle with sides a and b, since their hypotenuses make the same angles with either of the photon lines passing through a third event,E_3. Because the right triangles are similar, $a/b = c/d$ and therefore $ad = bc$.

The squared interval I^2 between events E_1 and E_2 is proportional to the area $(a-c)(b+d)$ of the rectangle of photon lines with the events at opposite vertices, which is just $ab - cd$, since $ad = bc$. But ab is proportional to the interval between the events E_1 and E_3 while cd is proportional to the interval between E_2 and E_3. Since E_1 and E_3 happen in the same place in Alice's frame, the squared interval between them is T^2, the square of Alice's time between them; since E_2 and E_3 happen at the same time in Alice's frame the squared interval between them is D^2, the square of Alice's distance between them. But since E_3 happens at the same time as E_1 and the same place as E_2 in Alice's frame, T and D are also Alice's time and distance between E_1 and E_2.

So $I^2 = T^2 - D^2$.