Universal Transition from Quasiperiodicity to Chaos in Dissipative Systems

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An exact renormalization-group transformation is developed which describes how the transition to chaos may occur in a universal manner if the frequency ratio in the quasiperiodic regime is held fixed. The principal low-frequency peaks in an experimental spectrum are universally determined at the transition. Our approach is a natural extension of Kolmogorov-Arnold-Moser theory to strong coupling.

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Our understanding of the onset of turbulence, within the context of low-order dynamical systems, has been extended from a qualitative description of various temporal regimes to a quantitative and universal set of predictions in the case of successive period doublings.[1,2] Although period doubling is not even the most prevalent route to chaos in low-aspect-ratio experiments, Feigenbaum's analysis has aroused great interest because the theory predicts that the Navier-Stokes equations are rigorously modeled by a one-dimensional map at the transition. His arguments are analogous to those used to describe scaling in critical phenomena. In this paper we show by renormalization-group methods how the transition from quasiperiodicity (flow with two incommensurate frequencies) to chaos can be made to proceed in a quantitatively universal manner.[3] This universality has not yet been seen experimentally, even though quasiperiodicity is a common percursor to turbulence, because it is associated with a critical point that can only be probed by varying two parameters in a consistent way.

On the mathematical side, our study suggests a means of realizing the strong-coupling limit of the small-divisor perturbation theory of Kolmogorov, Arnold, and Moser, who examined weakly nonlinear Hamiltonian and dissipative systems.[4] They realized that it is essential to work at a fixed frequency ratio or winding number which will become the second relevant variable at our fixed point. Conventional assumptions about the topology of the flows in function space generated by our renormalization group, we believe, imply the existence of a piecewise analytic variable change (a conjugacy), back to unperturbed quasiperiodic motion.[5] It is more interesting to study dissipative systems rather than Hamiltonian ones in this context because a fixed point found in one dimension is likely to carry over, rigorously, to an arbitrary number of dimensions just as one found for period doubling.[2]

Our calculations resemble Feigenbaum's treatment of period doubling in that we construct a transformation on a space of functions and find a nontrivial fixed point with the requisite linearized eigenvalues.[2] The alternative procedure, suggested by the Kolmogorov-Arnold-Moser proof, of successive variable changes is useful to keep in mind.[6]

The following map of an annulus illustrates succinctly the transition we propose to study.

Let

$$r_{i+1}-1=\lambda(r_i-1)-(a/2\pi)\sin(2\pi\varphi_i),\qquad \varphi_{i+1}=\varphi_i+\omega+r_{i+1}-1, \tag{1}$$

where (r,φ) are polar coordinates and $0\leq\lambda<1$. When $\lambda=1$ we obtain the "standard" area-preserving map of Chirikov.[4] Otherwise, since (1) contracts areas at a rate λ, there can be at most one invariant circle, $r=r(\varphi)$. Orbits on the invariant circle may have either a rational winding number $\rho(\omega,a)=p/q$ corresponding to mode locking in the differential system or an irrational ρ corresponding to a flow with two incommensurate frequencies. [In general $\rho=\lim_{i\to\infty}(\varphi_i-\varphi_0)/i$.] While a is analogous to the Reynolds number, the second relevant parameter ω should be thought of as a "bare" winding number that is adjusted to keep $\rho(\omega,a)$ fixed as a increases.

Since the nth iterate of (1) contracts areas at a uniform rate λ^n, this suggests that one can work with $\lambda = 0$. This defines the one-dimensional diffeomorphism,

$$\varphi_{i+1} = f(\varphi_i) = \varphi_i + \omega - (a/2\pi)\sin(2\pi\varphi_i). \qquad (2)$$

The renormalization group may then be used to justify the neglect of the radial contraction. The strong-coupling fixed point occurs at $a = 1$ where f has an inflection point. For $a > 1$, (2) is noninvertible and shows all the complexity of a one-dimensional map, while for $a < 1$ and almost all irrational winding numbers, the orbits are analytically conjugate to a simple rotation: $\varphi' = \varphi + \rho(\omega, a)$.[7]

We will consistently denote an irrational value of $\rho(\omega, a)$ by σ and write its continued-fraction representation as $1/[n_1 + 1/(n_2 + \ldots)]$. The lth rational approximate to σ, obtained by setting $n_i = 0$ for $i > l$, is p_l/q_l.

The universal features of the quasiperiodic-to-turbulent transition are restricted to low frequencies or long times so that the renormalization group is essentially functional composition. Our construction, however, will preserve both the character of f as a homeomorphism of the circle and its rotation number which will have important consequences when we consider spectra.

Both to define and to implement our renormalization group requires consideration of a larger class of functions than just analytic homeomorphisms of the circle. Specifically, let S_n be the class of pairs (ξ, η) of analytic homeomorphisms of the real line subject to the conditions (a) $\xi(\eta(0)) = \eta(\xi(0))$, $(\xi\eta)'(0) = (\eta\xi)'(0)$, (b) $\xi(0) = \eta(0) + 1$, (c) $0 < \xi(0) < 1$, (d) if $\xi'(0) = 0$ or $\eta'(0) = 0$ for $x \in [\eta(0), \xi(0)]$ then $\xi'(0) = \eta'(0) = \xi''(0) = \eta''(0) = 0$ but $\xi'''(0) \neq 0$ and $(\xi\eta)'''(0) = (\eta\xi)'''(0)$, (e) $\xi^n(\eta(0)) > 0$, $\xi^{n-1}(\eta(0)) < 0$ ($\xi\eta \ldots$, etc., denotes composition).

Define a mapping T_n on S_n by

$$T_n(\xi, \eta) = (\alpha\xi^{n-1}\eta\alpha^{-1}, \alpha\xi^{n-1}\eta\xi\alpha^{-1}), \qquad (3)$$

where $\alpha = 1/[\xi^{n-1}\eta(0) - \xi^n\eta(0)]$ and obeys $\alpha < -1$. The image under T_n of a member of S_n satisfies conditions (a)–(d), and in a neighborhood of the fixed point, (e) in addition.

Define a homeomorphism of the circle $f = f_{\xi,\eta}$ with $(\xi, \eta) \in S_n$ by identifying f with ξ on $[\eta(0), 0]$ and with η on $(0, \xi(0)]$ and denote the set of all such homeomorphisms by $\tilde{S}_n$. All analytic circle homeomorphisms [e.g., Eq. (2)] belong to $\tilde{S}_n$. The mapping T_n then induces a mapping $\tilde{T}_n$ on $\tilde{S}_n$ for which it may be proven that

$$\rho(\tilde{T}_n(f)) = 1/\rho(f) - n. \qquad (4)$$

$\tilde{T}_{n_1}$ then simply removes the first term in the continued fraction of $\sigma = \rho(f)$. Thus for any periodic continued fraction, $n_{i+s} = n_i$, $s \geq 1$, one could string together $T = T_{n_s} \circ \cdots \circ T_{n_2} \circ T_{n_1}$ and sensibly search for a fixed point of T. We will henceforth consider only $\sigma = \sigma_G = (5^{1/2} - 1)/2$ because then $n_i \equiv 1$ and $T = T_1$. The corresponding rational approximates are $p_l/q_l = F_{l-1}/F_l$ where F_l are Fibonacci numbers and satisfy $F_{l+1} = F_l + F_{l-1}$, $F_0 = 0$, $F_1 = 1$.

In this case, the fixed-point equations, $(\xi^*, \eta^*) = T(\xi^*, \eta^*)$, simplify considerably; in particular $\alpha = \xi^*(0)/[\xi^*(0) - 1]$. There is then a trivial fixed point corresponding to a pure rotation with $\xi^*(x) = x + \sigma_G$, $\alpha = -\sigma_G - 1 = -1/\sigma_G$, which has one unstable direction with eigenvalue $\delta = -\alpha^2$. The physical meaning of δ is clear from (4) since if $\rho(f) = F_{l-1}/F_l$ then $\rho(\tilde{T}f) = F_{l-2}/F_{l-1}$. In analogy with period doubling, δ measures the accumulation rate of the ω_l in (2) for which f has a F_{l-1}/F_l cycle. However, here ω_∞ corresponds to quasiperiodicity and not to incipient chaos.

At the nontrivial fixed point we can consistently assume that ξ^* and η^* are analytic functions of x^3. By retaining terms up to x^{36} and using a numerical technique suggested by Feigenbaum we find $\alpha = -1.288\,575$. Note that even if f is analytic, $\tilde{T}^k f$ is nonanalytic but continuous at the origin and end points of the unit interval. However, there is good numerical evidence that the conjugacy that relates $\tilde{T}^k f$ to any $\tilde{T}^l f$ is smooth for any f on the critical surface, i.e., $\lim_{l\to\infty} T^l f = f^*$. This fact, together with the existence of a fixed point f^*, establishes that spectra are universal in the sense discussed below.

Within the space of eigenfunctions, expandable in x^3 and consistent with conditions (a)–(e), we find one unstable eigenvalue $\delta = -2.833\,61$ with the same meaning as before. When arbitrary powers of x are allowed, we are able to show analytically from (a)–(e) that there is only one additional relevant mode with eigenvalue $\gamma = \alpha^2 = \sigma_G^{-\nu}$, which corresponds to the addition of a small linear term to the fixed point. (The relation $\gamma = \alpha^2$ is true for all winding numbers for which a fixed point exists.) Depending on its sign, the map either iterates into the chaotic regime where it initially has two critical points and is noninvertible, or remains invertible and iterates toward the trivial fixed point. In the space of all $f \in \tilde{S}$ with $\rho(f) = \sigma$ the nontrivial fixed point is a saddle and the trivial one a sink under $\tilde{T}$. If one assumes globally that there are no other fixed points then we conjecture that Kolmogorov-

Arnold-Moser–like results follow for all maps attracted to the trivial fixed point.

Another way of displaying the scaling behavior of (1) or (2) is to examine how points map back close to the origin after q_l iterations of f [n.b.: $p_l/q_l \to \sigma = \rho(f)$]. We therefore define

$$\zeta_l(x) = \alpha^l [f^{q_l}(\alpha^{-l}x) - p_l]. \quad (5)$$

For a suitable choice of α, $\lim_{l\to\infty}\zeta_l = \zeta$ exists as an analytic function on the line and to within a scale change is identical with ξ^*. The recursion formula for q_l generates nonlinear equations for ζ.[8]

We believe that a transformation like ours which preserves the circle homeomorphism structure and keeps track of the winding number is preferable to a straightforward iteration of f for the following reasons: (1) We can identify the universal numbers δ, γ, etc., as eigenvalues of T and make clear their physical meaning. (2) The minimal fixed-point functional equation is obvious, which, especially for $\sigma \neq \sigma_G$, is not clear otherwise. (3) There exist scaling equations for the conjugacy associated with $f^* = f_{\xi^*,\eta^*}$ which fix the experimental spectra. (4) It allows for mathematically rigorous proofs; e.g., Jonker and Rand have proven that the fixed-point and hyperbolic structure of T exist when ξ and η are analytic functions of $x|x|^{\epsilon}$, $0 \leq \epsilon \ll 1$.

Our renormalization group can readily be extended to higher dimensions and we believe that the fixed-point structure will remain unchanged.[9] Specifically, for annular maps [e.g., Eq. (1)], let $\Lambda(r,\varphi) = (\alpha^3 r, \alpha\varphi)$ and for $\sigma = \sigma_G$ define

$$T^{(2)}(E,F) = (\Lambda F \Lambda^{-1}, \Lambda F E \Lambda^{-1}),$$

where E and F are analytic homeomorphisms of the plane subject to conditions analogous to (a)–(e). A nontrivial fixed point of $T^{(2)}$ is $E^*(r,\varphi) = (0, \xi^*([r+\varphi^3]^{1/3}))$, $F^*(r,\varphi) = (0, \eta^*([r+\varphi^3]^{1/3}))$. Our *Ansatz* becomes plausible if one considers the family of curves along which the flow contracts exponentially onto the invariant curve, Γ. Below the critical point they intersect Γ transversally, but at the critical point after suitable iterations and rescalings they are tangent and converge to the family $r = -\varphi^3 + \text{const}$.

We will assume that our one-dimensional fixed point applies in higher dimensions and by invoking universality, analyze only the conjugacy h^* that reduces f^* to a pure rotation, i.e., $h^{*-1}f^* \times h^*(\theta) = \theta + \sigma_G$. The quantity of greatest physical interest is the periodic part of h^*, $\chi = h^* - \theta$, which is continuous but nondifferentiable. Its Fourier transform, $\tilde{\chi}(n)$, has prominent peaks at all the low-order Fibonacci series and scales as n^{-1}.[8]

If we use the natural range for h^* set by $\tilde{T}$, $1/(\alpha-1) \leq \varphi \leq \alpha/(\alpha-1)$, and allow $-\sigma_G^2 \leq \theta \leq \sigma_G$, then h^* is continuous on the interior of its domain and satisfies

$$h^*(\theta) = \begin{cases} \alpha^{-1}h^*(-\theta/\sigma_G), & -\sigma_G^2 \leq \theta \leq \sigma_G^3, \quad (6a) \\ f^{*-1}(h^*(\theta+\sigma_G-1)), & \sigma_G^3 \leq \theta < \sigma_G. \quad (6b) \end{cases}$$

Equation (6a) is a consequence of how $\tilde{T}$ acts on the invariant measure of any f with $\rho(f) = \sigma_G$ and may be generalized to other winding numbers. Once h^* is known on (σ_G^3, σ_G), (6a) recursively determines the rest of h^* on a sequence of intervals that converge onto the origin. The second equation smoothly relates h^* on $(-\sigma_G^4, \sigma_G^3)$ back to (σ_G^3, σ_G) so that (6a) and (6b) together should determine h^*.

To proceed further analytically it is convenient to use a piecewise linear approximation to f^* which utilizes a geometrically convergent set of vertices to approximate x^3 near the origin. The equation $Tf = f$ is satisfied everywhere. The corresponding value of α obeys $\alpha^6 + \alpha^5 - 1 = 0$, $\alpha = -1.2853$. The principal amplitudes in $\tilde{\chi}$ differ by no more than 10% from their exact values. Over the interval required by (6b), $f^{-1}(y) = (\alpha^2 + \alpha)y + 1/(1-\alpha)$. The relation $\tilde{\chi}(n) \sim n^{-1}$ for $n = m_1 F_i + m_2 F_{i-1}$, $i \gg 1$ follows rigorously.

The same scaling relation for $\tilde{\chi}(n)$ at the exact fixed point is proven from the lemma that any sufficiently smooth periodic function of h^* has a Fourier series bounded by η^{-2} for large η. The same lemma is used in the demonstration that spectra are universal. We conclude by suggesting how a laboratory experiment may be done to test the theory developed here.

In the quasiperiodic regime, an experimental spectrum is a series of peaks at all integer combinations of two incommensurate frequencies ω_1, ω_2. Any other choice of reference frequencies is related to (ω_1, ω_2) by an integer-valued matrix with determinant ± 1. Precisely the same condition is necessary and sufficient for the tails of the continued fractions of two irrational winding numbers σ and σ' to agree. Our renormalization group implies that any choice of (ω_1, ω_2) for a given experimental spectrum is associated with the same fixed point.

The easiest way to control the frequency ratio $\sigma = \omega_1/\omega_2$ in an experiment, as the Rayleigh number varies, is to introduce ω_2 by means of an external force. The optimal choice of σ experimentally is σ_G since it is the least susceptible to mode locking and one can expect to see the largest number of self-similar bands for a given level of noise.

There is a one-to-one relation between the low frequencies in the spectrum of a time series and $\tilde{\chi}$ which completely determines the former to within an overall scale. Specifically, the complex amplitude at $\omega = m_1\omega_1 - m_2\omega_2$, $0 \leq \omega < \omega_2$, is proportional to $\tilde{\chi}(m_1)$. The principal peaks in $\tilde{\chi}$ correspond to $(m_1, m_2) = (F_l, F_{l-1})$ $(\sigma = \sigma_G)$, and their power scales as σ_G^{-2l}. The condition that a given m be asymptotic is that it may be represented in the form $m_1F_l + m_2F_{l-1}$ with $l \gg 1$. An experimental determination of δ from the accumulation rate of periodic orbits will be difficult since to be useful it must distinguish between the trivial value of $\sigma_G^{-2} \sim 2.618\,03$ and 2.833 61.

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[8]This scheme as well as the spectra were examined numerically by Scott J. Shenker, to be published, and independently by ourselves.

[9]Strong evidence for this conjecture has been provided by Scott J. Shenker, who found that the critical indices of (1) and (2) agree; see Ref. 3.