Hysteresis Loop Critical Exponents in $6-\epsilon$ Dimensions

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The hysteresis loop in the zero-temperature random-field Ising model exhibits a critical point as the width of the disorder increases. Above six dimensions, the critical exponents of this transition, where the "infinite avalanche" first disappears, are described by mean-field theory. We expand the critical exponents about mean-field theory, in $6-\epsilon$ dimensions, to first order in ϵ. Despite $\epsilon=3$, the values obtained agree reasonably well with the numerical values in three dimensions.

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In a previous paper [1], we modeled hysteresis in magnetic and martensitic systems using the random-field Ising model at zero temperature. The model exhibited two features characteristic of these systems: the return-point memory effect and avalanche-generated noise. (The noise is called Barkhausen noise in magnetic systems and acoustic emission in martensites.) We also discovered a critical point, separating smooth hysteresis loops at large disorder where all avalanches are finite, from discontinuous hysteresis loops at small disorder where one avalanche turns over a fraction of the whole system.

Here we study this critical point in an expansion about mean-field theory. Figure 1(a) shows a schematic of the phase diagram for our model defined by Eq. (2) below. The vertical axis H is the external field. The horizontal axis R is the width of the probability distribution of the random fields f_i acting on each spin. The bold line represents the location $H_c(R)$ at which the infinite avalanche occurs, when the field $H(t)$ is adiabatically increasing from an initial state where all spins were pointing down. At small disorder, the first spin to flip easily pushes over its neighbors, and the transition happens in one burst (the infinite avalanche). At large enough disorder, the coupling between spins becomes negligible, and most spins flip by themselves: no infinite avalanche occurs. At a special value of the randomness $R=R_c$ the infinite avalanche disappears. We find a critical point with two relevant variables $r\equiv(R_c-R)/R_c$ and $h\equiv H-H_c(R_c)$ [1]. At this point we find a universal scaling law for the magnetization $m\equiv M-M_c(R_c)$

$$m\sim|r|^{\beta}\mathcal{M}_{\pm}(h/|r|^{\beta\delta}), \qquad (1)$$

where the $\pm$ refers to the sign of r.

We use a soft-spin version of the random field Ising model, whose energy at a given spin configuration $\{s_i\}$ is

$$\mathcal{H}=-\sum_{ij}J_{ij}s_is_j-\sum_i f_is_i+Hs_i-V(s_i), \qquad (2)$$

with the linear cusp potential $V(s_i)$ defined [2] through

$$V(s_i)=\begin{cases}(k/2)(s_i+1)^2 & \text{for } s<0,\\ (k/2)(s_i-1)^2 & \text{for } s>0.\end{cases}$$

Here, $k>0$ is the local curvature of the potential. The spins are coupled ferromagnetically by a nearest neighbor interaction $J_{ij}=J/z>0$, z being the coordination number of the lattice. We demand that $k/J>1$ to ensure stability of the system. H is a homogeneous external magnetic field; the f_i are randomly chosen from a Gaussian distribution $\rho(f)$ of standard deviation R. We study this system at zero temperature. It turns out for given H and f_i that there are many metastable states; which one of these the system picks depends entirely on its history

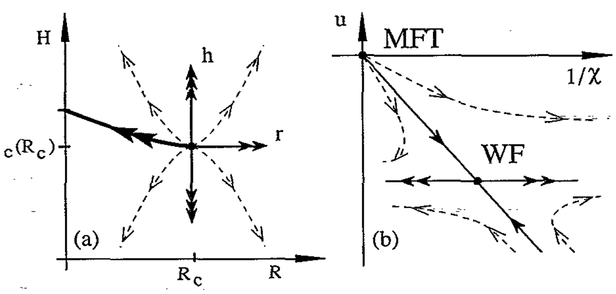

FIG. 1. Phase diagram and flows (schematic). (a) The vertical axis is the external field H, responsible for pulling the system from down to up. The horizontal axis is the width of the random-field distribution R. The bold line is $H_c(R)$, the location of the infinite avalanche (assuming an initial condition with all spins down and a slowly increasing external field). The critical point we study is the end point of the infinite avalanche line $(R_c, H_c(R_c))$. Using the analogy with the Ising model (see text) we also show the RG flows around the critical point. Here we ignore the RG motion of the critical point itself: equivalently, the figure can represent a section through the critical fixed point tangent to the two unstable eigenvectors (labeled h and r). Two systems on the same RG trajectory (dashed thin lines) have the same long-wavelength properties (correlation functions, etc.) except for an overall change in length scale, leading to the scaling collapse of Eq. (1). The r eigendirection to the left extends along the infinite avalanche line; to the right, we speculate that it lies along the percolation threshold for up spins (see Ref. [9]). (b) $O(\epsilon)$ RG flows below 6 dimensions in the (χ^{-1},u) plane (see text). Linearization around the Wilson-Fisher (WF) fixed point yields the exponents given to $O(\epsilon)$ in the table. In the vicinity of the repulsive $u=0=\chi^{-1}$ (MFT) fixed points one obtains the old mean-field exponents.

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(i.e., the way the external magnetic field H was varied at earlier times). We will study the history of a monotonically but adiabatically increasing external magnetic field. We impose purely relaxational dynamics, as defined by the equation of motion

$$\partial_t s_i(t) = -\frac{\delta(\mathcal{H})}{\delta s_i(t)}, \qquad (3)$$

where we have absorbed the friction constant into the definition of the time t.

(a) Formalism: We use the formalism of Martin, Siggia, and Rose [3] to rewrite the problem as a path integral for a generating functional Z, and then expand this functional about mean-field theory. This is done in analogy with the calculation for charge density waves (CDWs) by Narayan and Fisher [4,5]. We impose the dynamical equation (3) on the path integral at each time t by introducing it as a δ-function constraint using the well-known identity $2\pi\delta(f(s)) = \int_{-\infty}^{\infty} e^{i\hat{s}f(s)}d\hat{s}$:

$$Z \equiv \int [ds][d\hat{s}]J[s]\exp(S), \qquad (4)$$

where

$$S = \frac{i}{J}\int dt \sum_j \hat{s}_j(t)\Big(\partial_t s_j(t) - \sum_\ell J_{j\ell}s_\ell - H - f_j + \frac{\delta V}{\delta s_j}\Big). \qquad (5)$$

Derivatives of Z can be used to calculate the response functions and dynamic correlation functions for our model. $J[s]$ is a functional Jacobian, chosen such that Z is unity, independent of the f_j. This allows us to average over the disorder without fancy tricks (like replica theory).

We choose a particular regularization for the time integral. The simplest choice [4] is to require a force at time t to have an effect only after some time δt. That leaves us with $J[s] \equiv 1$. We now do an average over the random fields f_i, denoted by $\langle\,\rangle_f$, leading to the averaged generating functional

$$\bar{Z} = \int [ds][d\hat{s}]\langle \exp(S)\rangle_f. \qquad (6)$$

To expand about mean-field theory, we need change variables from s_j and $\hat{s}_j$ to the local fields $\eta_j = (1/J)\sum_\ell J_{j\ell}s_\ell$ at the sites (fluctuations about whose mean values we shall study). We do so by introducing another auxiliary field $\hat{\eta}_j$, and absorb a factor i in its definition, so

$$\bar{Z} = \int [d\eta][d\hat{\eta}]\prod_j \bar{Z}_j[\eta_j,\hat{\eta}_j] \times \exp\left\{-\int dt \sum_j \hat{\eta}_j(t)\left(\sum_\ell J^{-1}_{j\ell}J\eta_\ell(t)\right)\right\}, \qquad (7)$$

where $\bar{Z}_j[\eta_j,\hat{\eta}_j]$ is a local functional

$$\bar{Z}_j[\eta_j,\hat{\eta}_j] = \int [ds][d\hat{s}]\langle \exp S_j\rangle_f, \qquad (8)$$

and

$$S_j = \frac{1}{J}\int dt\left\{\sum_j J\hat{\eta}_j(t)s_j(t) + i\hat{s}_j(t)\left(\partial_t s_j(t) - J\eta_j - H - f_j + \frac{\delta V}{\delta s_j}\right)\right\}. \qquad (9)$$

We will now expand about the mean-field solution η_0 [the local field configuration about which the logarithm of the integrand in Eq. (7) is stationary]. Shifting the definition of η to $\eta-\eta_0$ so that $\langle\eta\rangle_f = 0$ leaves one with the generating functional

$$\bar{Z} = \int [d\eta][d\hat{\eta}]\exp(\tilde{S}) \qquad (10)$$

with an effective action

$$\tilde{S} = -\sum_{j,l}\int dt\, J^{-1}_{jl}J\hat{\eta}_j(t)\eta_l(t) + \sum_j\sum_{m,n=0}^{\infty}\frac{1}{m!n!}\int dt_1\cdots dt_{m+n}u_{mn}(t_1,\ldots,t_{m+n})\hat{\eta}_j(t_1)\cdots\hat{\eta}_j(t_m) \times\eta_j(t_{m+1})\cdots\eta_j(t_{m+n}). \qquad (11)$$

Here, the u_{mn} are the derivatives of $\ln \bar{Z}_i$ with respect to the fields $\hat{\eta}_j$ and η_j, and thus are equal to the local, connected responses and correlations in mean-field theory:

$$u_{m,n} = \frac{\partial}{\partial\epsilon(t_{m+1})}\cdots\frac{\partial}{\partial\epsilon(t_{m+n})}\langle s(t_1)\cdots s(t_m)\rangle_{f,l,c}. \qquad (12)$$

Local [4] (l) means that we do not vary the local field $(\eta_0)_j$ in the mean-field equation

$$\partial_t s_j(t) = J(\eta_0)_j(t) + H + f_j - \frac{\delta V}{\delta s_j(t)} + J\epsilon(t) \qquad (13)$$

when we perturb with the infinitesimal force $J\epsilon(t)$. The force $J\epsilon(t)$ is only allowed to increase with time, consistent with the history we have chosen. [For example, for $u_{1,1}(t,t')$ we add a force $J\epsilon\Theta(t-t_\prime)$ in Eq. (13), with $\Theta(t-t_\prime)$ being the step function, and take the derivative of $\langle s(t)\rangle_{f,l,c}$ with respect to ϵ and $t_\prime$.]

(b) Renormalization group (RG) treatment: We consider the $\hat{\eta}\eta$ term in the action (involving J_{jl}^{-1} and $u_{1,1}$) as the propagator in the RG treatment. Now we take some long-wavelength and low-frequency limits in analogy to [4,5]. For small wave vectors $J^{-1}(q) \sim 1/J + J_2 q^2$, and we rescale to give $J J_2 = 1$. We take the low-frequency part of the propagator, by Fourier transforming the $\hat{\eta}\eta$ term in time, expanding to first order in ω, and Fourier transforming back. The propagator (the $\hat{\eta}\eta$ term in the action) is thus (after rescaling)

$$-\int d^d q \int dt\, \hat{\eta}(-q,t)[-\partial_t + q^2 - \chi^{-1}/J]\eta(q,t)\,. \quad (14)$$

The bare value of χ is the static response, calculated in mean-field theory, to a monotonically increasing external magnetic field

$$\chi = 1/[2J^2\rho(-JM - H + k) - J(k-J)/k]\,, \quad (15)$$

where M is the magnetization at the external magnetic field H.

We use the Wilson-Fisher renormalization group transformation: In each step we integrate out modes of all frequencies and wave vectors within an infinitesimal wave vector shell [4]. We rescale through $x = bx'$, $t = b^z t'$. We choose the rescaling of the fields such that the q^2 term of the propagator and the $u_{2,0}$ term remain unchanged, since to first order in ϵ they have no loop corrections. Thus $\hat{\eta} = b^{-\frac{d}{2}-z}\hat{\eta}'$ and $\eta = b^{-\frac{d}{2}+2}\eta'$ [5]. Without loop corrections this implies $z = 2$. Keeping in mind that the $\partial/\partial\epsilon(t)$ in Eq. (12) rescale like b^{-z}, we arrive at $u'_{mn} = b^{[-(m+n)+2]d/2+2n}u_{mn}$. To lowest order in $\epsilon = d_c - d$, the only relevant terms are those which do not flow to zero under rescaling at the upper critical dimension d_c (we will see that $d_c = 6$ for our critical point). $u_{1,0}$ is trivially zero because we expand around the stationary point. The $u_{1,2}$ term, in the static limit, has bare value $w = -2J^2\rho'(-JM-H+k)$: it becomes relevant for $d < 8$. The $u_{1,3}$ term starts at $u = 2J^3\rho''(-JM-H+k)$ in the static limit, and is relevant for $d < 6$. Finally, the $u_{2,0}$ term stays marginal. In the static limit that we consider $u_{2,0}$ can be treated as a constant. We are left with the effective action

$$\tilde{S} = -\int d^d q \int dt\, \hat{\eta}(-q,t)[-\partial_t + q^2 - \chi^{-1}/J]\,\eta(q,t) + \sum_j \Big[(1/2)\int dt\, \hat{\eta}_j(t)[\eta_j(t)]^2 w + (1/6)\int dt\, \hat{\eta}_j(t)[\eta_j(t)]^3 u$$
$$+ (1/2)\int dt_1 \int dt_2\, \hat{\eta}_j(t_1)\hat{\eta}_j(t_2) u_{2,0}(t_1,t_2)\Big]\,. \quad (16)$$

Our ϵ expansion can be applied not only to the critical point $(R_c, H_c(R_c))$, but to the entire line $H_c(R)$ at which the infinite avalanche occurs. In mean-field theory, the approach to this line is continuous, with a power-law divergence of the susceptibility χ and precursor avalanches of all scales. Above 8 dimensions the action is purely quadratic at the fixed point, and the infinite avalanche line [where $1/\chi = 0$ and $w = -2J^2\rho'(-JM-H+k) \neq 0$] presumably remains critical. For $d = 8 - \tilde{\epsilon}$, Fig. 2(a) shows the correction to vertex w to first order in $\tilde{\epsilon}$. The incoming lines at a vertex stand for η operators, and the outgoing lines are $\hat{\eta}$ operators. The low-frequency form of the propagator is approximately $\delta(t-t')$ [4] but we have to observe causality; an example of a diagram forbidden by causality is given in Fig. 2(b). Applying the usual approximations [6], we obtain for the recursion relation for w to $O(\epsilon)$:

$$w'/2 = b^{(-d/2+4)}\Big\{ w/2 + (u_{2,0}/2)(w/2)^3 8/(4\pi)^4 \times \int_{\Lambda/b}^{\Lambda} dq (q^2 - \chi^{-1}/J)^{-4} \Big\}\,. \quad (17)$$

Writing $b^{(-d/2+4)} = b^{(\tilde{\epsilon}/2)} = 1 + (\tilde{\epsilon}/2)\ln b$ and performing the integral over the momentum shell $\Lambda/b < q < \Lambda$ leaves us with the recursion relation:

$$w'/2 = w/2 + (w/2)[\tilde{\epsilon}/2 + u_{2,0}(w/2)^2 4/(4\pi)^4 \ln b]\,. \quad (18)$$

Since $u_{2,0} > 0$ this means that for $\tilde{\epsilon} > 0$ there are only two fixed points with $w' = w$: either $w = 0$, which we will discuss in the next paragraph, or $w = \infty$. We see

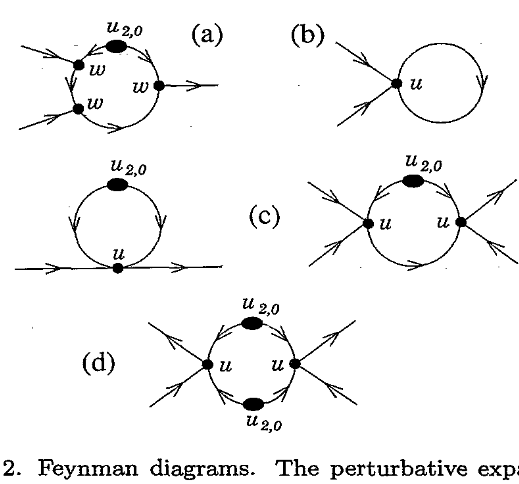

FIG. 2. Feynman diagrams. The perturbative expansion about mean-field theory is presented here by Feynman diagrams. (a) The correction to $O(\tilde{\epsilon})$ to the vertex w in an expansion about 8 dimensions, see Eq. (18) in the text. (b) An example of a diagram forbidden by causality. (c) The relevant corrections to first order in $\epsilon = 6 - d$ for the constant part χ^{-1}/J in the propagator and for u. Using the same techniques that lead to Eq. (18) we find the following recursion relations: $(\chi^{-1}/J)' = b^2[\chi^{-1}/J + u_{2,0}u/(4\pi)^3\Lambda^2(1 - 1/b^2)/4 + u_{2,0}u/(4\pi)^3(\chi^{-1}/J)\ln b]$ and $u' = u + u[\epsilon + 3/(4\pi)^3 \times u_{2,0}u]\ln b$. $u_{2,0}$ does not get any loop corrections of $O(\epsilon)$. (d) An example of a correction to the vertex $u_{2,2}$ which contributes only to $O(\epsilon^2)$, which is not present in the regular Ising model (or the thermal random-field Ising model).

TABLE I. Universal exponents for critical behavior in hysteresis loops. The exponents β and δ tell how the magnetization scales with r and h, respectively, Eq. (1). ν is the correlation length exponent, measured (numerically) using finite-size scaling.

Exponents	ϵ expansion with $\epsilon = d-6$, at $\epsilon = 3$	Simulation [1] in 3 dimensions
$1/\nu$	$2-\epsilon/3=1$	1.0 ± 0.1
β	$0.5-\epsilon/6=0$	0.17 ± 0.07
$\beta\delta$	$1.5+O(\epsilon^2)=1.5$	2.0 ± 0.3
δ	$3+\epsilon=6$	(around 12)

that under the recursion relation (18) any system that has a bare value $w \neq 0$ when $1/\chi = 0$ will flow to the fixed point $w = \infty$. We interpret this as indication that the transition is a first-order transition for $d < 8$. Indeed, in three dimensions the simulation showed a first-order transition without critical fluctuations for these systems.

The critical point we are interested in here is the fixed point where $w = 0$. At $d = 6$ the first nonquadratic contribution u becomes relevant; i.e., the upper critical dimension [7] for the critical end point is 6. We now compute, to $O(\epsilon)$, the corrections to the recursion relations. The relevant diagrams are shown in Fig. 2(c). Figure 1(b) shows the corresponding RG flows in the (χ^{-1}, u) plane.

The loop corrections look very similar to the loop corrections in the usual Ising model in $d-2$ dimensions. In fact, to $O(\epsilon)$, they are the same. This can be seen either by direct computation (see caption Fig. 2) or by noticing that the Feynman rules for our diagrams are the same as those for the finite-temperature random-field Ising model [8] [except that we have extra vertices which are irrelevant to $O(\epsilon)$]. This latter model has been mapped to all orders in ϵ onto the regular Ising model, using supersymmetry and other arguments. This analogy tells us that to $O(\epsilon)$ we get the same RG flows (Fig. 1) and the same corrections to our exponents as one finds in the usual Ising model in $d-2$ dimensions [6] (see Table I).

This mapping does not extend to the next (ϵ^2) term in the series: Fig. 2(d) shows a correction of $O(\epsilon^2)$ to the vertex $u_{2,2}$, which then contributes in $O(\epsilon^2)$ to the propagator. This is comforting, as otherwise the critical properties of our model in $d = 3$ would have completely mapped onto the $d = 1$ thermal (nonrandom) Ising model, which has no finite-temperature phase transition at all. Indeed, this was a substantive concern for the thermal random-field Ising model, which despite the correspondence above was proven to have a transition in $d = 3$: the ϵ expansion for that model summed over physically incorrect metastable states. By controlling the history of the external field (as in [4,5]), we have been careful to specify the particular metastable state in our calculations.

The ϵ expansion for our model is technically much simpler than that for other disordered extended dynamical systems: e.g., interface [9] or charge-density-wave [4] depinning, where an infinite family of relevant operators made necessary a functional renormalization group. The relative ease of our calculation may make possible further extensions: calculating the corrections to the equations of state, calculating the history-dependent critical behavior, or addressing the avalanche distributions.

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