The onset of chaos: Full renormalization-group calculation

(Sethna, “Entropy, Order Parameters, and Complexity”, ex. 12.XXX)

In this exercise, we implement Feigenbaum’s numerical scheme for finding high-precision values of the universal constants

$\alpha = -2.50290787509589282228390287322$

$\delta = 4.66920160910299067185320382158$,

that quantify the scaling properties of the period-doubling route to chaos (Fig. 12.17}, Exercise ‘Period doubling’). This extends the lowest-order calculation of the companion Exercise ‘The onset of chaos: Lowest order renormalization-group for period doubling’}.

```
αFeigenbaum = -2.502907875095892822283902873218;
δFeigenbaum = 4.669201609102990671853203821578;
```

Our renormalization group operation (Exercises ‘Period doubling and the renormalization group’ and the companion Exercise) coarse-grains in time taking g→g∘g, and then rescales distance x by a factor of α. Centering our functions at x=0, this leads to

$T[g](x) = \alpha\, g(g(x/\alpha))$.

We shall solve for the properties at the onset of chaos by analyzing our function-space renormalization-group by expanding our functions in a power series

$g(x) \approx 1 + \sum^{N}_{n=1} G_n\, x^{2n}$.

Notice that we only keep even powers of x; the fixed point is known to be symmetric about the maximum, and the unstable mode responsible for the exponent δ will also be symmetric.

```
(* N us a reserved variable; use Nn instead *)
g[Nn_][x_] := 1 + Sum[ ..., {n, 1, Nn}]
T[g_][x_] := α ...
(* We'll also want the derivative of g later *)
Dg[Nn_][x_] := Sum[ ..., { ...}]
(* Test your functions by plotting
  them.G=[-1.5,0,0,...] should give T[g] close to g *)
G0[1] := -3/2
G0[n_] := 0 /; n ≠ 1
Plot[{g[2][x], T[g[2]][x] /. {α → 1/g[2][1]}} /. G → G0, {x, 0, 2}]
```

First, we must approximate the fixed point $g^*(x)$ and the corresponding value of the universal constant α. At order N, we must solve for α and the N polynomial coefficients G^*_n. We can use the N+1 equations fixing the function at equally spaced points in the positive unit interval:

$T[g*]\,(x_m) = g^*(x_m)$, $x_m = m/N$, m={0,…,N}.

We can use the first of these equations to solve for α.

(a) Show that the equation for m=0 sets $\alpha = 1/g^*(1)$.

We can use a root-finding routine to solve for G^*_n.

(b) Implement the other N constraint equations above in a form appropriate for your method of finding roots of nonlinear equations, substituting your value for α from part (a). Check that your routine at N=1 gives values for $\alpha \approx -2.5$ and $G_*1 \approx -1.5$. (These should reproduce the values from the companion Exercise part (c).)

```
Nmax = 20;
 For[Nn = 1, Nn <= Nmax, Nn = Nn + 1,
   xm = Range[...];
   vars = Table[{G[n], G0[n]}, {...}];
   eqns = Table[... == ... /. α → 1/g[Nn][1], {x, xm}];
   GStar[Nn] = FindRoot[..., WorkingPrecision → 50];
   α[Nn] =  ...] /. GStar[Nn]]
Table[{Nn,  ...}, {Nn, 1, Nmax}] // MatrixForm
Table[{Nn, αFeigenbaum - ...
```

Now we need to solve for the renormalization group flows T[g], linearized about the fixed point $g(x)=g^*(x)+\epsilon\psi(x)$. Feigenbaum tells us that $T[g^*+\epsilon\psi]=T[g_*]+\epsilon\mathcal{L}[\psi]$, where $\mathcal{L}$ is the linear operator taking $\psi(x)$ to

$$\mathcal{L}[\psi](x) = \alpha\, \psi(g^*(x/\alpha)) + \alpha\, g^{*\prime}(g(x/\alpha))\, \psi(x/\alpha).$$

(d) Derive the equation above.

ANSWER HERE

We want to find eigenfunctions that satisfy $\mathcal{L}[\psi]=\lambda\psi$. Again, we can expand $\psi(x)$ in a polynomial

$$\psi(x) = \sum^{N-1}{}_{n=0}\, \psi_n\, x^{2n} \qquad (\psi_0 \equiv 1).$$

We then approximate the action of $\mathcal{L}$ on ψ by its action at N points x_i, that need not be the same as the N points x_m we used to find g_*. We shall use $x_i = (i-1)/(N-1)$, i=1,…,N. (For N=1, we use $x_1=0$.) This leads us to a linear system of N equations for the coefficients ψ_n, using the previous two equations.

$$\sum^{N-1}{}_{n=0}\, [\alpha\, g(x_i/\alpha)^{2n}+\alpha\, g'(g(x_i/\alpha))\, (x_i/\alpha)^{2n}]\, \psi_n = \lambda \sum^{N-1}{}_{n=0}\, x^{2n}{}_i\, \psi_n$$

These equations for the coefficients ψ_n of the eigenfunctions of ▯ is in the form of a generalized eigenvalue problem

$$\sum^{N-1}{}_{n=0}\, L_{in}\, \psi_n = \lambda \sum^{N-1}{}_{n=0}\, X_{in}\, \psi_n.$$

The solution to the generalized eigenvalue problem can be found from the eigenvalues of X^{-1}L, but most eigenvalue routines provide a more efficient and accurate option for directly solving the generalized equation given L and X.

(e) Write a routine that calculates the matrices L and X implicitly defined by the previous two equations. For N=1 you should generate 1×1 matrices. For N=1, what is your prediction for δ? (These should reproduce the values from the companion Exercise part (d).)

```
(* Make sure your matrix hasn't transposed rows (i)
 and columns (n).Each row should give powers of one x_i. *)
(* Avoid 0^0 for N=1 by using 'Evaluate' to set up n=0 column in X *)
xtildes[Nn_] := Range[0, 1, 1/(Nn - 1)]
xtildes[1] := {0.}
X[Nn_] := Table[Evaluate[Table[ ..., {n, 0, Nn - 1}]], {xtilde, xtildes[Nn]}]
X[1]
X[3]

L[Nn_] :=
 Table[Evaluate[Table[α[Nn] g[ ...] ^ ( ...) + α[Nn] Dg[ ...] ( ...) ^ ( ...) /. GStar[Nn],
    {n, 0, Nn - 1}]], {xtilde, xtildes[Nn]}]
L[1]
L[3] // MatrixForm
Eigenvalues[{L[3], X[3]}]

Nmax = 20;
For[Nn = 1, Nn <= Nmax, Nn = Nn + 1,
 eigvals =  ...[{L[Nn], X[Nn]}];
 δ[Nn] = eigvals[[1]]]
Table[ ...] //  ...
Table[ ...] //  ...
```