Lecture 1.2

Localization and the Integer Quantum Hall effect

The aim of this lecture is to explain how disorder – which creates a random potential for electrons, thereby destroying translational symmetry – is a necessary and sufficient condition for the observed plateaus, for noninteracting electrons. So long as we are ignoring interactions, the integer quantized Hall effect reduces to the study of localization and transport – as in [P636 notes, Lec. 2.4]and [P636 notes, Lec. 2.5]– but in a large magnetic field, which changes all the answers. Specifically, we review Anderson localization, Landauer conduction, and percolation theory; these had all become mature subjects just prior to the discovery of the QHE. Some of the notions turned out to be misleading for the case of large field, others were quite handy.

1.2 A Anderson localization (review)

The key ideas of the IQHE refer to localization as a function of energy. The δ-function Landau Level (LL) peaks in the DOS are broadened by disorder into bands. Only the state at the middle is delocalized and carries current. As you change the magnetic field, you change the Landau level spacing and the filling factor, hence the Fermi level will move relative to the LL peaks even though the electron density is held fixed by the doping. So long as it's delocalized, you see σ_{xy} quantized and $\sigma_{xx} = 0$.

A key concept is "localization": that is when, due to disorder, *electrons don't diffuse* away from a starting point. You know of "Anderson localization" which occurs in zero magnetic field (reviewed in Sec. **??**, below.) Localization in very large B field is qualitatively different. It turns out there is only one critical energy at which there is a delocalized eigenstate, and it is carrying all the current in a one-dimensional fashion.

Anderson model of localization

Start with a lattice of sites i, each of which has one orbital $|i\rangle$ which span the Hilbert space of the model. The Hamiltonian is

$$\mathcal{H} = -\sum_{\text{n.n. } i,j} t|i\rangle\langle j| + \sum_i V(i)|i\rangle\langle| \tag{1.2.1}$$

The first term is a lattice version of the kinetic energy operator, $-(\hbar^2/2m)\nabla^2$; the bandwith is $\sim 6t$ in $d = 3$. The second term is a random potential, with $\langle(\Delta V)^2\rangle \sim W^2$.

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You could have studied localization in a continuum model, using the familiar K.E. operator and a continuous potential $V(\mathbf{r})$. To make the correspondence, let the lattice spacing a be the separation of $\mathbf{r}_1$ and $\mathbf{r}_2$ such that $V(\mathbf{r}_1)$ and $V(\mathbf{r}_2)$ are mostly uncorrelated. Then let $t \sim \hbar^2/2ma^2$, the K.E. associated with the wavefunction varying over a distance a.

Localization length

An eigenstate is defined to be localized if

$$|\psi(\mathbf{r})|^2 \sim \exp(-|\mathbf{r}|/\xi_L) \tag{1.2.2}$$

This is meant only in the sense of an average over the ensemble of possible random potentials $\{V(i)\}$; or as the *envelope* of a particular $\psi(\mathbf{r})$. Inside the envelope, $\psi(\mathbf{r})$ likely has rapid oscillations (at the Fermi wavevector of the electron gas) as well as irregular extinctions and renewals of strength. The opposite of "localized" is extended, meaning the envelope of $|\psi(\mathbf{r})|^2$ is constant througout the system (and $\xi_L \to \infty$.) The states exhibiting "weak localization" (see [P636 notes, Lec. 2.2]) are extended states.

In (1.2), xi_L is the "localization length". Just as the correlation length specifies how far correlations propagate in a system with an order parameter, ξ_L tells far an electron propagates in a localized system. (We could define this more precisely but we'd need Green's functions.)

Mobility edge

Notice that if any eigenstate is extended at energy E, all other states at E can be mixed with it, so the property of being extended or localized is a function of E; similarly the localization length depends on energy, $\xi_L(E)$. The general picture is given in Fig. 1.2.1. There are intervals of localized and extended states; the energy of the border between them is the "mobility edge" E_c (so called by analogy to the band edge). If the Fermi level falls in the extended states, the material is a metal, whereas if it falls in the localized states we have an Anderson insulator.

Critical phenomena occur near the mobility edge, $\xi_L(E)\ |E - E_c|^{-\nu_L}$, where ν_L is the localization length exponent. For $d = 3$, the exponent is still very roughly known, $\nu_L \approx 1.6$ [1] The exponent is *universal*: it depends on spatial dimension and on the symmetries of the problem, but not on any other details.

The picture in Fig. 1.2.1 is for $d = 3$, where, if W/t is small enough, we get extended eigenstates. In $d = 2$, *all* states are localized. Of course, if $W/t \ll 1$, then ξ_L becomes extremely large. If your system size is smaller than ξ_L, an eigenstate connects from one side to the other (and can thus conduct), so the system is extended for practical purposes.

Large B is different

Localization in a magnetic field is different (it breaks the time-reversal symmetry). We'll find that not (or not quite!) all the states are localized. This would seem to be a bit of a paradox. In the pure case – no scattering – in zero field an electron travels

[1] K. Slevin and T. Ohtsuki, Phys. Rev. Lett. 82,382 (1999); S. Waffenschmidt, C. Pfleiderer, and H. von Löhneysen, Phys. Rev. Lett. 83, 3005 (1999). try A. Roemer and Michael Schreiber, pp. 3-19, "The Anderson Transition and its Ramifications-Localisation, Quantum Interference, and Interactions", in 'Lecture Notes in Physics' , ed. T. Brandes and S. Kettemann, (Springer, Berlin, 2003).

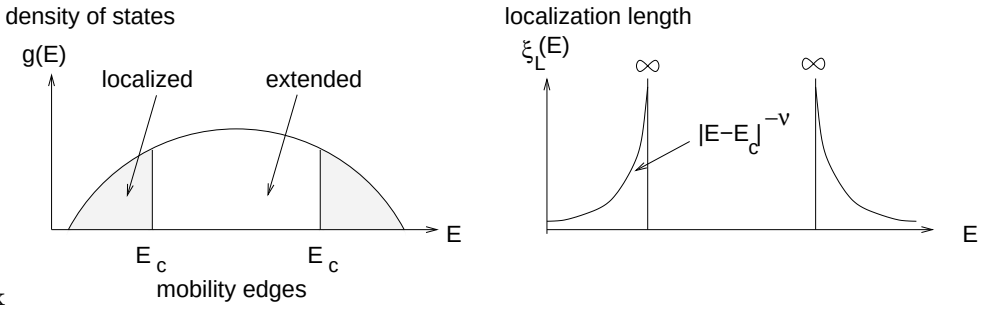

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Figure 1.2.1: Anderson localization in $d = 3$: (a). schematic density of states, showing the mobility edge between extended and localized states. (This picture corresponds to $t \sim W$, and the bandwidth is of the same order.) (b). critical divergence of localization length, as a function of eigenstate energy (c). [not included here](A lecture on Anderson Loc. would also show a graph of the diffusion constant as a function of E: it is nonzero only in the (middle) interval of extended states, and goes to zero (with a critical exponent) at the mobility edges.)

ballistically; but in a magnetic field it is localized (in the sense of band theory): its band structure is flat (at $(n + \frac{1}{2})\hbar\omega_c$ for any wavevector) and its group velocity is zero. In the presence of disorder, in the energies near the band edges (where disorder localizes states, field or no field), the B field might indeed increase the degree of localization.

Near the band center, another competing effect is more important. [2] Absent the field, states tended to be localized due to weak localization effects – the constructive interference of two different backscattering paths (related by time reversal symmetry) augmenting the backscattering. (These are responsible for the localization with a slow onset we get for weak disorder in $d = 2$). Namely, the field randomizes the Aharonov-Bohm phases so the interference will now have a relative random phase, and there is no more constructive interference between the two backscattering paths. This was responsible for the negative magnetoresistance... described in [P636 notes, Lec. 2.2].

In the next sections, I want to flesh out the localization idea with a geometrical picture of the actual eigenstates, which are spread out over equipotential lines. The particular ones that matter are *edge states*, which run along the edge of the sample (or of the "Hall droplet", a connected portion which is filled with electrons exhbiting a quantum Hall effect.) In section 1.2 D it is related to a percolation-like picture of the localization transition in large field limit.

1.2 B Landauer approach: Edge state picture of the IQHE

The Landauer approach is a scattering approach, which is a completely different way of thinking than (say) the Boltzmann equation, or the Kubo formula. (Whereas the Kubo formula may lend itself better to averaging over disorder, Landauer lends itself better to describing a particular device.) The idea is, to evaluate the current, we shoot an electron into our system and ask, where does it end up? (Or rather, with what probability does it it arrive at various end states.) If it comes out the opposite terminal, it contributed

[2] See T. Dröse, M. Batsch, I. Kh. Zharekeshev, and B. Kramer, "Phase diagram of localization in a magnetic field", *Phys. Rev.* B 57, 37 (1998)

to the current, but not if it backscatters and comes back out the same terminal it went in.

Review one-dimensional wires

Let's first review picture in zero field. (See [P636 notes, Lec. 2.5].) We can have one-dimensional wires by a narrow strip of semiconductor, or by bending graphene (carbon sheets) to form a nanotube, so the transverse confinement creates subbands. In effect, each subband has its own dispersion $E_n(k)$. Imagine that $E_n(k)$ crosses the Fermi level for just one of the flavors; this will be called "one channel" for transport: think of it as a highway with one lane going in each direction. (Perhaps even better, a railway line with a track each way, or a pair of conveyor belts.) All the action will be near the two Fermi points, $+k_F$ and $-k_F$. The electrons have velocities $+v_F$ and $-v_F$ respectively, so I will call them "plus movers" and 'minus movers" (the usual term is "right movers" and "left movers", but the channel will be oriented in the y direction shortly).

Label the terminals 1 and 2. We have two possible incoming states, $|1+\rangle$ (so called because it's in terminal 1 and a plus mover) or $|2-\rangle$; the outgoing states are $|1-\rangle$ and $2+\rangle$. We can write a scattering matrix $\mathcal{S}_{ij} \equiv$ amplitude for scattering to outgoing i, from incoming j. Quite generally this can be written

$$\mathcal{S} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} r & t \\ t & -r \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \tag{1.2.3}$$

where $e^{i\alpha}$ and $e^{i\beta}$ are phases factored out to make t, r real, and the exact form here is required (it turns out) by time-reversal symmetry. Here t is the transmission amplitude – heading straight through on the plus-moving track and r is the reflection amplitude, so $|t|^2 + |r|^2 = 1$.

The main assumption of Landauer theory is that each terminal is connected to a reservoir, which has has an equilibrium Fermi distribution, and injects electrons into that lead with that distribution. The effect of a voltage difference, $V = V_1 - V_2 > 0$, is that the reservoirs have different Fermi energies: $E_{F1} - E_{F2} = -eV$. Hence, the injected right-movers have a smaller Fermi wavevector (hence a lower density) than left-movers.

Consider the ideal case that $|t| = 1$ (and, for simplicity, zero temperature), i.e. every plus mover injected at terminal 1 reaches terminal 2 and vice versa. The density (per length) of extra electrons is

$$n_- - n_+ = \frac{1}{2\pi}\Delta k_F = \frac{1}{2\pi}\frac{eV}{|dE(k)/dk}. \tag{1.2.4}$$

Now notice $dE/dk \equiv \hbar v_g$, where v_g is the group velovity at k_F, i.e. the Fermi velocity v_F. There are $v_F n_+ v_F$ plus movers per unit time passing a given point, and $v_F n_- v_F$ minus movers, so the net current is

$$I = (-e)(n_+ - n_-)v_F = \frac{e^2}{2\pi\hbar}V = R_Q^{-1}V. \tag{1.2.5}$$

where $R_Q = h/e^2$ is the quantum of resistance. Thus, a single channel has a (two-terminal) quantized conductance $G = R_Q^{-1}$.

If we had $N_c > 1$ channels (i.e., that many of them cross the Fermi energy), then $G = N_c R_Q^{-1}$. But if we have one channel with some backscattering, so $|t|^2 < 1$, we get a conductance $G = |t|^2 R_Q^{-1}$: the quantization is *not* robust to 1 part in 10^9.

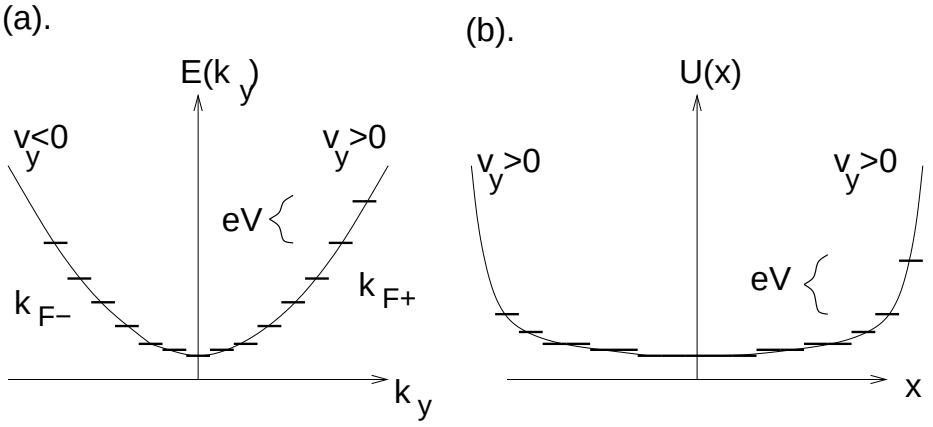

Figure 1.2.2: (a). Dispersion relation of a one-dimensional channel extending in the y direction, showing the occupied states extending up to different Fermi levels for the the $+$ movers and $-$ movers. Here V denotes the voltage difference between the two leads, injecting electrons in opposite directions. (b). Eigenstates of a Hall system in large B field, with a potential well $U(x)$, showing a different Fermi level for the two edges, which have a potential drop V between them.

Quantized Hall case

We can now describe the QHE in a Landauer framework. stringy states combined with the Landauer description of [3] Recall (Lec. [1.1]) that (if we limit states to the LLL) each eigenstate is a strip along a contour: a one-dimensional channel carrying current, as we discussed above in Landauer theory. The big difference is that here, instead of having a family of different eigenfunctions in a channel with different k vectors, electrons now move big difference: electrons move in only one direction. (If the Landauer channel was like a two-lane highway, this is like a one-way street!) These are called "edge states" since the ones which can be filled at small cost are on the edge of a region with filled states.

Let's examine the comparison in more detail. We saw x and y are conjugate in LLL. That is, in the quantum Hall system, the x position *is* the k_y momentum. Compare the two parts of Fig. 1.2.2.

We obtain $I_y = R_Q^{-1} V_x$ in exactly the same way as before, except in place of the index k (i.e. k_y), we have an index x, and instead of the group velocity proportional to $dE(k)/dk$, we get the drift velocity proportional to $dU(x)/dx$. In other words, we got the Hall conductance to be quantized!

$$G_{xy} = R_Q^{-1} \tag{1.2.6}$$

That was too easy. The physically non trivial fact is $|t| = 1$ is now generic. The reason is that in the Hall system, the plus and minus movers are physically segregated between the two edges. To increase I_y, we must increase the number of plus movers, and thus must increase the Fermi level on side where the plus movers.

[3] K. von Klitzing, *Physica* B 184, 1 (1993); The Landauer approach is also the centerpiece of Johnson & Kirczenow (1997), handout [in P636, 1999]. Compare also C. L. Kane and M. P. A. Fisher in Das Sarma and Pinczuk (1997).

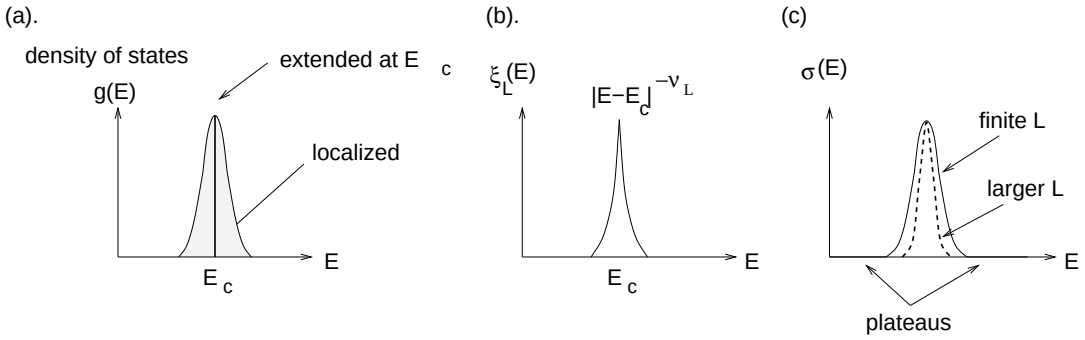

Figure 1.2.9: (a). Density of states of a Landau level; extended states exist at a unique critical energy. (b). Localization length; the critical exponent ν_L is defined the same way as in the Anderson model, but has a different universal value for the case with a magnetic field. (c). longitudinal conductance $\sigma_{xx}(E)$, if we could vary the Fermi energy, for samples of small finite width L.

1.2 C Experimental measurement of localization length

We now go back to the ideas of the [first] section, but holding on to the notion that backscattering is the cause of resistance. It turns out (as shall be justified in the following section) that for localization in a large magnetic field, there is an extended state but only at one energy E_c in the middle of each Landau level. The localization length is defined in the same way, by (1.2.2), but its critical exponent ν_L takes a different value.

We can measure it as follows. Consider a sample of finite width L, so this is the separation between the edge states moving in opposite directions. When L is comparable to ξ_L, the states extended as far as the other side of the sample, so they might as well be genuinely extended states over the entire interval that $\xi_L(E) > L$. It follows that the width of the E interval where we see extended behavior is $\sim L^{1/\nu_L}$; thus by measuring the broadening of this peak on samples made with a variety of L's, we can infer ν_L.

There are other ways to do it using the temperature broadening effect.

In the present case, the idea is that the two opposite edge states are separated by L, and each extends (in a convoluted way) an average distance of ξ_L from the edge. So, if $\xi_L > L$, it touches the far edge, and scattering is possible between the two edge states. That causes nonzero σ_{xx} and a non ideal value of σ_{yx}.