Lecture 1.1

Introduction to QHE: inversion layers and Landau Levels

*Basic phenomena*

The quantized Hall effect was originally observed in 1980. A “two-dimensional electron gas” was made by doping the semiconductor GaAs in a specially clean fashion. Turn on a big (transverse) magnetic field $B$. Measure and plot the Hall resistivity $\rho_{xy}$ as a function of $B$.

At higher temperatures we see the straight line we expected from (say) Drude theory. But at the lowest $T$, we get a bunch of plateaus, (Fig. ??) on each of which the Hall resistivity behaves as

$$\rho_{xy} = -\frac{1}{\nu} R_0$$

(1.1.1)

where

$$\frac{1}{\nu}$$

is a rational fraction, and

$$R_0 \equiv \frac{h}{e^2} = 25.812807 \text{k}\Omega$$

(1.1.2)

is the quantum of resistance.

At not-so-low $T$, of a few degrees, the visible plateaux all have $\nu = \text{integer}$, which is called the “integer quantum Hall effect” (IQHE). The “fractional quantum Hall effect” (FQHE) discovered a couple years later, shows up $T \sim 0.1 \text{K}$; this will be discussed in Lec. 2.1 and later lectures. (Most properly, it should be called quantized Hall effect, but I won’t be consistent.)

Furthermore, within a plateau, the longitudinal resistivity vanishes $\rho_{xx} = 0$. These statements about the resistivity tensor are exact only at $T = 0$, but the plateaux can be made repeatable to within a part in $10^9$, so QHE is the de facto definition of the ohm, and consequently has great practical significance to the national laboratories that maintain units around the world. (the formal definition of ohm still depends on the kilogram).

Part 1 of these lectures will treat the IQHE and the role of disorder; this can all be understood in the framework of non-interacting electrons. Part 2 is about the FQHE, which depends on interactions. The most important bit is Laughlin’s wavefunction and the Laughlin quasiparticles, gapped elementary excitations which have fractional charge and fractional statistics. I’ll also discuss the (gapless) edge quasiparticles which

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Figure 1.1.1: The Hall plateaus (schematic). The dashed line shows the Drude prediction. A few fractional plateaus are shown too.

are responsible for transport, and allow experimental tests of the fractional charge, as well as the composite fermion theory which (going beyond the Laughlin states) explains all the FQHE plateaux.

Philosophy of this module (2006)

One purpose of this course is to teach some techniques that are widely applicable in condensed-matter theory. That is less true of this module: many techniques of quantized Hall effect are useful mainly within that subject. Hence, this will be more a survey of the phenomena, at the lowest technical level needed to honestly represent the physics. When we do encounter a general method – such as coherent states for semiclassical dynamics, in this lecture – I’ll point it out. There will also be abundant opportunities to review subjects in transport, magnetism, and (perhaps) superconductivity.

I apologize for the raw nature of this lecture, and no doubt some of the rest: this is the first time I have written it down at full length. I hope there are no serious errors or factors of 2, but you may encounter inconsistencies in notation, and (especially) redundancies where the same thing is said twice, due to a lack of time to combine or eliminate the duplicates. You may sometimes find a snatch of text from an obsolete presentation in the version I used for P 636 in 1998, due to incomplete editing. Finally, some sections may be sparse, not much more than a transcription of what I wrote on the board.

Bibliography about the QHE

There are a great many reviews; I’ll mention all of them that actually seem useful. The basic entry point to the QHE is Prange and Girvin (1990)\footnote{R. E. Prange and S. M. Girvin, ed., The Quantum Hall Effect, 2nd ed. (Springer, Berlin, 1990). A decent capsule introduction of the same vintage is in the first three sections of “The Quantum Hall Effect” by J. R. Schrieffer and D. P. Arovas, in Frontiers and Borderlines in Many-Particle Physics, ed. R. A. Broglia and J. R. Schrieffer (Proceedings of the International School of Physics “Enrico Fermi”, 1987).} a collection of articles on the basics of the IQHE and FQHE as known in the 1980s.

The other fundamental references that brings it up to date is Das Sarma and Pinczuk (1997)\footnote{S. Das Sarma and A. Pinczuk, ed., Perspectives in Quantized Hall Effects, Wiley, New York, 1997).} but you can’t just jump in – this book assumes you have the background at the
level of Prange and Girvin (1990). A relatively contemporary review, which emphasizes the approaches to the FQHE, is "The Quantum Hall Effect: The Article." ³

1.1 A Two-dimensional electron gas: preliminaries

This section reviews the essentials of realizing the two-dimensional electron gas (2DEG) and its behavior in a magnetic field.

Realization of jellium

"Jellium" means replacing the periodic background of positive ions by a uniform background charge (See [P636 notes, Lec. 1.7].) A rather good two-dimensional realization of jellium is found in a semiconductor along the interface between two different compositions (both intrinsic); say GaAs in AlAs. This is grown by molecular beam epitaxy (MBE) so the interface is atomically flat. The difference in bandgaps produces an effective step in the potential felt by electrons, i.e. a one-dimensional confining square well of width \( W \) (see Fig. 0.1.2). Also, a layer of positively charged dopant ions (usually Si, which has a charge +1 in a III-V semiconductor) is set back from the interface. This fixes the electron concentration.

The charge density of ions varies spatially at the scale of the typical dopant spacing, which must also be the typical electron spacing \( a_s \) in the layer. Then from basic electrostatics, the variations in the resulting electric potential decay as \( e^{-z/a_s} \) with the setback distance \( z \). Thus if \( z \) is a few times \( a_s \), this potential is essentially uniform, realizing the jellium model. Scattering by the potential is then negligible and the electron mobilities can be very large. ⁴

There are variations on this; the original samples were silicon MOSFETs, where the electron density \( n \) is controlled by a gate. There could be AlAs on one side and bulk GaAs on the other; then the electric field potential plus the potential step suffice to make a confining well. What matters is that the dopant layer is physically separate, so we get the doping without the scattering.

Basic energy scales

See Table [1.0]. As with any new system, one’s first approach is to list the important scales, and to see which ones are physically important. In the present case, it makes sense to frame them as energies, since the most important experimental condition is low \( T \) which freezes them out.

The confinement energy due to the well, i.e. the gap to the next subband, is \( E_{\text{box}} \sim 10^3 \text{K} \gg T \): hence only the lowest subband is involved and our electron gas is strictly two-dimensional. Also, the density \( n \) implies \( E_F \sim 60 \text{K} \gg T \), so (until we turn on the large field) it is a degenerate Fermi gas. The scattering lifetime uncertainty \( \hbar/\tau_{\text{sc}} \sim T \), so we won’t ignore the disorder. (From most viewpoints, the disorder is extremely small, but for our purposes it is just barely small enough.) This is also the magnitude of the perturbation which splits the degeneracy of the Landau levels, as shown in Fig. 0.1.2 (b).


⁴Of course, the electron spacing in this jellium is far larger than in an elemental metal. On the other hand, the effective Bohr constant in a semiconductor is \( a_B^* = e_0 \hbar^2/m_e c^2 \), which is also larger (\( \sim 100a_B \) in Si or Ge).
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Figure 1.1.2: (a). Geometry of an inversion layer; the potential (bottom) combines the effects of the difference in bandgaps and the electric field due to a distant layer of Si$^+$ donors. (b). Density of states in large field: Landau levels are broadened by disorder.

Two basic scales are both ways to express the magnitude of magnetic field as a time or a length: the cyclotron frequency

$$\omega_c = \frac{eB}{m^*_e c}; \quad (1.1.3)$$

and the “magnetic length” $\ell_B$ defined by

$$\frac{1}{\ell_B^2} = \frac{eB}{\hbar c}. \quad (1.1.4)$$

In a $B$ field the electron eigenstates (without a potential) are Landau levels separated by $\hbar\omega_c$. In the large fields (10’s of tesla) available, we have $\hbar\omega_c = 100K \gg T$. This is crucial: it means all the action is in one Landau level (depending on which one is partially filled, which depends on the ratio of electron density to magnetic field or “filling factor” $\nu$). Note that achieving large fields is the most fundamental experimental limitation – unlike all the other parameters, available fields have not changed by orders of magnitude over the decades.

Another list of conditions needed is

$$\omega_c\tau_{sc} \gg 1; \quad (1.1.5)$$

$$E_{int} \approx \frac{e^2}{\epsilon r_{cc}} > T; \quad (1.1.6)$$

$$E_{int} > \langle (\Delta V(r))^2 \rangle \approx \frac{\hbar}{\tau_{sc}}. \quad (1.1.7)$$

The last two conditions, on electron-electron Coulomb interactions, are more appropriate for the FQHE.
1.1 B. MACROSCOPIC DESCRIPTION

If we had bare electrons, the Zeeman splitting between spin states would be comparable to $\hbar \omega_c$ and frozen out of the problem. So until further notice, we shall assume the electron spins are all up — in effect, spinless — even though the reality is more complicated.

Hamiltonian?

We’ll adopt a spinless free electron/jellium model, with effective mass $m_*$ (remember this is considerably smaller than a bare electron mass). The Hamiltonian is

$$\mathcal{H} = \frac{|p|^2}{2m_*}$$

with momentum

$$p = p_0 + \frac{e}{\hbar c} A,$$

where $p_0$ is the momentum operator in absence of a field; and a compensating charge background is assumed.

The problem has a complete translational and rotational (around $z$) symmetry, but this is unavoidably obscured by the vector potential. Either of the important gauge choices has its advantages. In this section we adopt Landau gauge, which keeps us as much translational symmetry as possible. In sec. ?? we’ll adopt the symmetric gauge which keeps rotational symmetry instead.

In Landau gauge the vector potential is

$$A = B(0 - x)$$

where as noted already, our convention is $B = -B\hat{z}$.

1.1 B Macroscopic description

The experiment

A schematic figure is shown in Fig. 0.1.1. (Papers usually plot $\rho_{xy}$ versus $B$, here I plot the inverse of each.)

It is only at $T = 0$ that $\sigma_{xx} = 0$ exactly in the plateaus. Experiments show a nonzero $\sigma_{xx} \sim e^{-\Delta/T}$; the activated dependence (like the conductivity in an intrinsic semiconductor) shows there is a gap $\Delta$.

Let me pause to comment on gaps and gaplessness. This gap, and the gap between Landau levels, is crucial to the IQHE. (The same thing is true for FQHE except the origin of that gap is interactions.) On the other hand, there is another sense in which we have gaplessness. Namely, the Landau levels have a massive degeneracy; an electron in one of these states can get to another with zero excitation energy, so in that sense it is a gapless problem. More importantly, the degeneracy means every perturbation is a singular one. There is a rich variety of correlated states that can be built out of linear combinations of the Landau level states, at no cost but gaining energy from the perturbing term.

Conductivity tensor

In the elementary Drude picture, the Hall conductivity is

$$\sigma_{xy} = \frac{ne\tau}{B}$$
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![Schematic picture of an irregular sample with a voltage V placed between contacts at r₀ and r₁. Dotted lines are local electric field; solid lines are both equipotentials and also current flow, when \( \sigma_{xx} = 0 \) and \( \sigma_{xy} \neq 0 \). The total Hall conductance is exactly \( \sigma_{xy} \) regardless of shape.](image)

Figure 1.1.3: Schematic picture of an irregular sample with a voltage \( V \) placed between contacts at \( r_0 \) and \( r_1 \). Dotted lines are local electric field; solid lines are both equipotentials and also current flow, when \( \sigma_{xx} = 0 \) and \( \sigma_{xy} \neq 0 \). The total Hall conductance is exactly \( \sigma_{xy} \) regardless of shape.

The conductance is independent of anything but the density \( n \). Using the definitions, this can be massaged into the form

\[
\sigma_{xy} = \frac{e^2}{h} \cdot \frac{hc}{e} \cdot \frac{n}{B} = \frac{1}{R_0} \cdot \frac{\Phi_0 n}{B} = \frac{\nu}{R_0}
\]  

(1.1.12)

The filling factor \( \nu \) can also be written as

\[
\nu = \frac{n}{n_B}, \quad \text{where} \quad n_B = \frac{B}{\Phi_0}
\]  

(1.1.13)

In other words, \( n_B \) is the density of flux quanta, and \( \nu \) is the number of electrons per flux quantum. So (??) says, if \( \nu \) happens to be exactly an integer, Drude predicts the correct quantized value. Indeed, these points are the centers of the plateaus. The mystery is why it is stuck on this value when electrons are added or taken away.

The fact that zero resistivity corresponds to zero conductivity is only a superficial paradox. Write the \( 2 \times 2 \) conductivity tensor

\[
\sigma = \begin{pmatrix} 0 & \sigma_{xy} \\ -\sigma_{xy} & 0 \end{pmatrix}
\]  

(1.1.14)

so the resistivity tensor is

\[
\rho \equiv \sigma^{-1} = \begin{pmatrix} 0 & \sigma_{yx}^{-1} \\ \sigma_{xy}^{-1} & 0 \end{pmatrix}
\]  

(1.1.15)

You can also see, for nonzero \( \sigma_{xx} \ll \sigma_{xy} \), we will have \( \rho_{xx} \approx \sigma_{xy}^{-2} \sigma_{xx} \), so the resistivity is proportional to conductivity (keeping in mind that the off-diagonal element is stuck at, or very near to, the quantized value.)

Why two dimensions is special

For a start, it is conductance that relates to a combination \( R_0^{-1} \) of fundamental constants, and only in \( d = 2 \) does conductivity have the same units as conductance. But that’s not enough: even if conductivity were precisely quantized, how could we measure it, if it depended on details of the sample shape? Indeed, how could it even be defined to one part in \( 10^9 \) – moving a single atom changes the conductance (at least, the ordinary conductance) much more than that.
Consider a rectangular sample of dimensions \( L_x \times L_y \), with (say) electric field in the \( x \) direction. Converting from current density and field to current and voltage, we find \( I_x = \frac{L_y}{L_x} \sigma_x V_x \), i.e. the conductance is proportional to the aspect ratio. But for the Hall conductance, the same dimension \( L_x \) enters the conversion of both current and voltage, and cancels: \( I_y = \sigma_{yx} V_x \) irrespective of the dimensions: the Hall conductance (which is what one can actually measure) is the Hall conductivity.

In fact the conductance is independent of sample shape, no matter how irregular. Consider the picture in Fig. 0.1.3: the lines of current now (being normal to the field) follows equipotentials, and they both spread out the same way (since current density is proportional to local field). We have a voltage

\[
V = \int_{r_0}^{r_1} dr \cdot E(r)
\]  

integrating along any line we like. On the other hand, the current passing \( I_y \) passing across that line is

\[
I_y = \int_{r_0}^{r_1} J(r) \cdot \hat{n}(r) dr
\]  

here \( \hat{n}(r) \) is the local normal to the line. But \( \hat{n}(r) dr = \hat{z} \times dr \), and \( J(r) = \sigma_{yx} (\hat{z} \times E(r)) \) by assumption. Thus

\[
I_y = \int_{r_0}^{r_1} \sigma_{yx} (\hat{z} \times E(r)) \cdot (\hat{z} \times dr) = \int_{r_0}^{r_1} \sigma_{yx} E(r) \cdot dr = \sigma_{yx} V.
\]

Later on, we’ll encounter a couple other ways in which \( d = 2 \) is central. For one thing, as we are about to discover, the \( x \) and \( y \) coordinates function as a canonically conjugate pair – obviously the same cannot be true with a third component. Secondly, we will take advantage of compact ways of writing the electron wavefunctions in one Landau level using the complex variable \( z = x + iy \), which obviously can be one only on the plane.

1.1 C Classical motion (review)

The classical behavior of an electron in a strong magnetic field gives clues for understanding the quantum eigenstates in the next sections. (In particular, what happens in
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We'll find the laws of motion are literally “screwy” in that they have a screw sense due to the massive violation of time reversal symmetry (of the large $B$ field). Like a gyroscope, the response to a push is to move sideways.

Let’s consider the general behavior with a constant force, $f_0$, and $B = B_z \hat{z}$.

Including the Lorentz force in Newton’s law of dynamics gives

$$f_0 + \frac{q}{c} (v(t) \times B)_{||} = m \frac{d}{dt} v(t). \quad (1.1.19)$$

Notice that here, $f_0$ and $v(t)$ are assumed to be confined in the plane perpendicular to $\hat{z}$, and “$\ldots_{||}$” denotes projection into that plane. Any motion of an electron out of that plane will be stymied by the restoring force of the 2DEG confining potential, which would be messy to treat explicitly here.

As a digression, what would happen if (say) $B$ is parallel to the $xy$ plane? So long as there isn’t room in the confinement for even the smallest cyclotron orbit, i.e. $W_{\text{box}} < \ell_B$, the effect is small; the electron’s motion (orbital degree of freedom) depends only on the perpendicular component $B_z$. On the other hand, the electron’s spin degree of freedom will be strongly affected (equally strongly, if there is no spin-orbit coupling). Thus, tilting the magnetic field is an important experimental trick to separate effects of Zeeman coupling from the orbital effects (that are our current business.)

**Special case: no constant force**

First consider $f_0 = 0$. Of course, the electrons move in cyclotron orbits with cyclotron frequency $\omega_c$ given by (0.1.3). The equations of motion are mathematically identical to simple harmonic motion, but the two variables oscillating $\pi/2$ out of phase are $v_x$ and $v_y$, or equivalently $x$ and $y$, rather than $x$ and $p_x$. The size of the $n$-th quantized orbit is $\sim \sqrt{n} \ell_B$, where $\ell_B$ is given by (0.1.4); the $n$-th Landau level is built (see below) from those states.

The oscillator behavior foreshadows the fact that, for low energy states in big fields, $\sqrt{2ke}/\ell_B$ and $\sqrt{2\hbar y}/\ell_B$ behave as canonically conjugate variables. One way to make this more definite is to note that low energy dictates that $|p|$ is small in (0.1.8) and thus (in symmetric gauge) $p_x \approx -eBy/2c = -\hbar y/2\ell_B^2$.

**Special case II: constant drift**

We can get uniform motion $v(t) = \dot{v}$ under a constant force, at the unique in-plane $\dot{v}$ at which $f_0$ cancels the Lorentz force. Thus

$$\frac{q}{c} \dot{v} \times (B_z \hat{z}) = -f_0 \quad (1.1.20)$$

For an electron $q = -e$, and (note!) we take the convention henceforth that

$$B = -B \hat{z} \quad (1.1.21)$$

The minus sign gives a counterclockwise drift inside a well, and (not independent, but more important) means we’ll encounter analytic (rather than anti-analytic) eigenfunctions in the symmetric gauge (Sec. 0.1 E).
Including both terms

We have found the homogeneous and inhomogenous terms of the differential equation (0.1.19); now put them together. We get

\[ v(t) = \bar{v}(t) + R\omega_x(\cos \omega_c t, -\sin \omega_c t) \]  

(1.1.22)

The electron’s path is a circular orbit (at frequency \( \omega_c \)) about around a “guiding center” \( \bar{r} \), which drifts normal to the force \(-\nabla V\) at a velocity proportional to it

\[ \frac{d\bar{r}}{dt} = \left( \frac{e}{\epsilon B} \right) \hat{z} \times \nabla V(r) \]  

(1.1.23)

The small fast orbits are clockwise, the drift is counterclockwise around a smooth well (see Fig. 0.1.4).

Conjugacy

Eq. 0.1.23 says

\[ \frac{d}{dt} \bar{x} \propto \frac{\partial V}{\partial \bar{y}} \]  

(1.1.24)

\[ \frac{d}{dt} \bar{y} \propto -\frac{\partial V}{\partial \bar{x}} \]  

(1.1.25)

These are the equations of two canonically conjugate variables! (This works so long as \( B \) is big enough that the orbit radius is smaller than the scale on which \( V(r) \) varies.)