Lecture 5.5

Spin waves

There are several reasons we want to know the elementary excitations of an ordered magnet: (i) experiments measure them (ii) they determine thermal behaviors (at low $T$) such as the specific heat or the reduction of the order parameter by thermal fluctuations (iii) the quantum-mechanical ground state (and its correlations) are expressed in terms of them.

Spin waves are the analog for magnetically ordered systems of lattice waves in solid systems; and just as a quantized lattice wave is called a “phonon”, a quantized spin wave is called a “magnon”. \(^1\) Spin waves are the Goldstone mode corresponding to spin rotation symmetry. acquire a gap in the case of violations of that symmetry, i.e. of anisotropy. If a continuous symmetry remains (e.g. in the easy plane anisotropy ferromagnet), one expects a gapless mode still exists.

The chief business of this lecture is to derive the spin-wave modes either by linearizing the spins’ classical equations of motion or else by putting their quantum mechanical Hamiltonian in quadratic form. There are several applications of magnons. First, their zero-point motions at $T = 0$ define a quantum ground state. Fluctuations (whether at $T = 0$ or due to thermal excitation) also reduce the net moment, analogous to the reduction of the Bragg peak in a solid (“Debye-Waller factor”) by vibrations. If the moment reduction is larger than the spin’s length itself, as in some frustrated antiferromagnets, it suggests that the true ground state lacks spin order. Finally, magnons dominate the low temperature specific heat of the spins. Because ferromagnets and antiferromagnets have different power laws in their dispersion relations, and because of the possibility of a gap, there is greater variety in the results than in the phonon case.

5.5 O Preview: approaches to spin waves

We assume a general translationally invariant exchange Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{ij} J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j.$$  \hspace{1cm} (5.5.1)

\(^1\)Another sort of “spin wave” excitation occurs when a large magnetic field is imposed on quantum liquids with spin (Fermi liquids – namely alkali metals or liquid $^3$He – as well as cold, dilute Bose gases with a spin degree of freedom). There is a magnetic resonance in this field; its frequency has some dispersion with $q$, so we can have traveling excitations. It is not a Goldstone mode of any ordered state, and always has a gap.
where \( J_{ij} \equiv J(r_i - r_j) \). We want to turn it into \( E_0 + \mathcal{H}_{\text{magnon}} \), where

\[
\mathcal{H}_{\text{magnon}} = \sum_q \hbar \omega(q) \left( \frac{1}{2} + b_q^\dagger b_q \right)
\]

(5.5.2)

There are at least five paths to the harmonic spin wave spectrum – all of which give the same results.

1. “Classical (or “semiclassical”) approach

This is convenient for visualizing a spin wave in space. It is the exact analog of the classical approach to phonons in elementary solid state physics. First you find the classical lattice vibrations, and then you quantize them as harmonic oscillators. This approach is the simplest and is presented in Sec. ??, below.

2. “Hydrodynamic” classical approach

Let us recollect the analogous approach in the phonon case. The speed of sound (coefficient in \( \omega(q) \approx c(q) \) for small \( |q| \)) depends only on macroscopic parameters – the density and the elastic moduli – and this is sufficient if we only care about the long-wavelength (“hydrodynamic”) dynamics.

A spin-wave mode is also a Goldstone mode and hence has a hydrodynamic description too. (That’s liable to be a starting point in field theory, which often aims at coarse-grained descriptions that are independent of the underlying lattice.) The elastic moduli and density in a solid correspond to the spin stiffness \( \rho_s \) (see Modern Models Lec. 5.3 ) and the inverse susceptibility in an antiferromagnet. \(^2\)

3. Hopping approach.

This approach emphasizes the discrete nature of spins. It is convenient if the ground state wavefunction is just a single spin state; that is the case in a ferromagnet, but not an antiferromagnet. Just take the ground state, flip one spin, and investigate how it hops quantum-mechanically Since spin is conserved, the only possible other states are those with one flipped spin on some other lattice site: thus it reduces to the problem of a particle hopping on a lattice. The single-particle eigenstates must then be Bloch states and the dispersion is computed just like the band structure of a hopping electron (as done in Modern Models Lec. 1.1 ). This excitation is a magnon. \(^3\)

When you flip two spins, the Hamiltonian is not the same as two free particles – they have an effective interaction. If one views the excitations as harmonic oscillators, this interaction is anharmonicity, and it can cause scattering just as for phonons. At sufficiently low energies (meaning long wavelengths in the case of phonons or spinwaves), the dominant effect of such anharmonicities is just to modify the prefactor in \( \omega \propto |q|^\frac{3}{2} \) or \( |q| \).

The hopping approach does not generalize to the antiferromagnet: even if you start with the fully ordered Néel (coherent states) state, which alternates completely up and completely down spins, successive exchanges can get you to any state with net \( S_z = 0 \), including lots of states which look nothing like the Néel state.

\(^2\)See Principles of Condensed Matter Physics by P. Chaikin and T. C. Lubensky for a thorough introduction to hydrodynamic approaches.

\(^3\)The hopping approach is detailed in Ashcroft & Mermin, pp. 704-709.
4. Holstein-Primakoff approach

The most general quantum-mechanical way to set up spin waves uses the “Holstein-Primakoff transformation”. That is an algebraic way to represent the spin operators for a spin in terms of a harmonic oscillator raising/lowering operator (or equivalently, boson creation/annihilation operator). This is worked out in Sec. 5.5 B. It gives a systematic expansion in terms labeled with powers of the small parameter $1/S$.

The main advantage of this approach is that this expansion includes further anharmonic terms with (e.g.) three or four magnon operators. Such terms are necessary to set up the kinetic theory of magnons, since they tell the magnon-magnon scattering or spontaneous decay of magnons. Anharmonic terms also enter the renormalization – the modification of the spin-wave velocity due to quantum fluctuations, which is a big effect for small $S$. (Finally, the anharmonic terms are also necessary in some order-by-disorder calculations – compare Sec. 5.5 Z – if the harmonic term fails to resolve all degeneracies.)

5. Spin deviation operators

This is a streamlined, more physical implementation of the Holstein-Primakoff approach. It replaces the operators $a_i$ and $a_i^+$ by real linear combinations of them, $\sigma_x$ and $\sigma_y$. These (to linearized order) are just the spin deviations, as in the classical dynamics; they correspond to the position and momentum of a harmonic oscillator. (The difference is that they have the same dimensions; indeed, when the magnetic configuration is collinear, $\sigma_x$ and $\sigma_y$ are related by the symmetry of rotating around the spin direction.) The harmonic-order part of the Hamiltonian is quadratic in (the Fourier transforms of) $\sigma_x$ and $\sigma_y$, so we can directly construct the harmonic oscillator raising and lowering operators: no Bogoliubov diagonalization is necessary. Sec. 5.5 C gives the details.

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There is a further advantage from this way of thinking. The number of spins per magnetic unit cell, $m$, is always greater than one – possibly large, when the spin structure is non-collinear. In the Holstein-Primakoff approach, people often define $m$ distinct flavors of boson, one on each site of the magnetic cell’s basis, and the spin-wave modes are eigenvectors of an $m \times m$ matrix. But so long as the spin configuration is collinear or coplanar, we can let one spin deviation component be normal to this plane, and represent the other component in terms of the small angle that the spin rotates about this axis. In these terms, the equations are often the same on each site, i.e. the spin waves can be represented in terms of the original cell; the matrix to be diagonalized has a smaller dimension, and the symmetries of the modes are more transparent.

6. Path integral

This approach is found in Sec. 11.1 of Interacting Electrons and Quantum Magnetism, by A. Auerbach.

5.5 A Preview of results

I will give the resulting spectrum first, and then derive it. Recall we have an ordered state – for simplicity a collinear one, so (without loss of generality) $(\hat{b}_i) = \eta_i \hat{S}_z^i$, where

$$\eta_i = \exp iQ \cdot r_i$$

(5.5.3)
define the sign of the spin on site $i$ with respect to the ordering axis; and $Q$ is the ordering wavevector, which is typically on the boundary of the lattice Brillouin zone.

Now let
$$\tilde{J}(q) \equiv \sum_{R} J(R)e^{i\mathbf{q}\cdot\mathbf{R}}$$
and then define
$$\tilde{K}(q) \equiv \tilde{J}(Q) - \tilde{J}(q).$$
Recall the “local field” on site $i$ is defined as
$$h^{(0)}_i = \sum_{j} J_{ij}s_j;$$

For a simple magnetic configuration such that all sites are equivalent, observe that $\tilde{J}(Q) = h^{(0)}$. Note also that, for the simple two-sublattice antiferromagnet, $2Q$ is always a reciprocal lattice vector for the lattice (of spins), so $q + 2Q \equiv q$.

Then the spin-wave dispersion is given by
$$\omega(q)^2 = \tilde{K}(q)\tilde{K}(q + Q).$$

What does this dispersion look like? At small $q$, we trust $\tilde{K}(q) \sim |q|^2$ since it is analytic and symmetric under $q \leftrightarrow -q$. The coefficient(s) are just the spin stiffness(es) that couple to angle gradients in the elastic theory: in the case of a high-symmetry (cubic, square, triangular) lattice, there is just one stiffness $\rho_s$ and
$$\tilde{K}(q) \approx \rho_s |q|^2$$

(\text{NOTE: this is probably wrong by factors of $S$.)}

**Ferromagnet**

A simple ferromagnet with all spins parallel just has $Q = 0$ (hence $m = 1$) so we get essentially identical equations for $\sigma_{q_x}$ and $\sigma_{q_y}$. The frequency is
$$\omega(q) = S\tilde{K}(q)$$
and thus, for small $q$,
$$\omega(q) \approx \frac{1}{2}D|q|^2$$

\text{Need to fix this better... There is just one (circular) polarization mode possible for each $q$ in the ferromagnet case. That agrees with the number of degrees of freedom going into the calculation: two linearized transverse components from each spin.}

**Antiferromagnet**

On the other hand, in the antiferromagnet the first factor in (5.5.7) scales as $|q|^2$ while the second factor goes to a constant as $q \to 0$. Indeed,
$$\tilde{K}(Q) = \chi^{-1},$$
where $\chi$ is the susceptibility for making a uniform magnetization. Hence the generic dispersion for an antiferromagnet is
$$\omega(q) = c|q|$$
with \( c = \sqrt{\rho_s / \lambda} \).

There are two polarizations of each long-wavelength mode in the antiferromagnet case (we could pair \( \sigma_{qx} \) and \( \sigma_{q+y} \) or \( \sigma_{qy} \) and \( \sigma_{q+x} \). (Recall the magnetic Brillouin zone is halved so the modes at \( q \) and at \( q + Q \) are actually the same.) In the end, we correctly get one mode for each allowed \( q \) vector of the original Brillouin zone.

**Experiments**

Spin wave dispersions are measured by magnetic (inelastic) neutron scattering. This gives richer data than the Bragg scattering – in principle, an entire curve \( \omega(q) \) as function of wavevector. Hence, fitting to the spin wave dispersion is the commonest way to experimentally determine the exchange constants (or other coefficients in the spin Hamiltonian).

### 5.5 B Holstein-Primakoff setup

The Holstein-Primako transformation expresses the spin operators in terms of Bose (oscillator) operators as follows:

\[
S_+ = (2S - a^\dagger a)^{1/2}a \\
S_- = a^\dagger(2S - a^\dagger a)^{1/2} \\
S_z = S - a^\dagger a
\]

with the Bose operators obeying \([a, a^\dagger] = 1\) as usual. In other words, the spin raising operator is just the oscillator raising operator but its coefficients depends on the value of \( S_z \).

One thing to watch out for: while the spin’s Hilbert space is \((2S + 1)\) dimensional, the oscillator’s Hilbert space is infinite dimensional: it includes an unphysical subspace with \( S_z = -S - 1, -S - 2, \ldots \). However, the coefficient in (5.5.13) and (5.5.14) is zero for the step connecting \( S_z = -S \) and \(-S - 1\); hence, the unphysical subspace has no matrix elements connecting it to the physical subspace with \( S_z = S, \ldots, -S \).

**Spin wave derivation**

Let’s apply the Holstein-Primakoff transformation to an interacting system with spin order. For the state space of spin \( i \), we use a quantization axis oriented along the classical direction \( \mathbf{n}_i \). We imagine the typical excitations on the ladder are small compared to our large parameter \( S \). So we Taylor expand the square-roots in (5.5.13) and (5.5.14), collecting powers of \( a, a^\dagger \).

Our series is actually an expansion in powers of \( 1/S \); to sort its terms, note that each power of \( a \) or \( a^\dagger \) is \( O(1) \). (This follows from the uncertainty relation, since their commutator is 1.) Note that, after we collect terms to make a polynomial in the operators, each operator term doesn’t have a single number as a coefficient but rather a whole series in \( 1/S \). For a first approximation, it is generally safe to only keep the dominant term of this coefficient (largest power of \( S \)).

The \( O(S) \) terms in (5.5.15), or the constant terms in powers of \( a, a^\dagger \), combine to give the classical energy (which is \( O(JS^2) \)). The terms linear in \( a, a^\dagger \) (as you can check)

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4 Notice that the bottom state of the boson ladder is the spin-up state, since we think of excitations as reducing the moment from the original value.

5 Being careful with commutators: \((a^\dagger a)^2 = a^\dagger a a^\dagger a \neq a^\dagger a^2 a^2\)
always cancel: their coefficients are of the form \( e_\pm \cdot \mathbf{h} \), where \( e_\pm \) are vectors in spin space for the spin deviations, and \( \mathbf{h} \) is the local field. But \( \mathbf{h} \parallel \mathbf{n} \) e in a classical ground state, and \( \mathbf{n} \cdot e_\pm = 0 \) by definition of the small spin deviations.

The next nonzero term in the expansion is the \( O(JS) \) term, which is quadratic in \( a \) and \( a^\dagger \). This can be manipulated – eventually – into the desired form of a harmonic oscillator Hamiltonian, (5.5.2).

**Bogoliubov transformation?**

In the customary Holstein-Primako approach it’s nontrivial to get the harmonic oscillator form, since some of the quadratic terms are two creation or two annihilation operators, e.g. \( \hat{a}^\dagger \hat{a}^\dagger \) and \( \hat{a} \hat{a} \). The remedy is to set up a Bogoliubov transformation in which we write \( b \) as a linear combination of \( \hat{a}^\dagger \) and \( \hat{a} \), in such a fashion that the coefficients of \( b^\dagger b^\dagger \) and \( bb \) terms all cancel. Finally, the cubic and higher terms in powers of \( a \) are converted into cubic and higher powers of \( b \) operators, which correspond to interactions or scatterings of spin waves.

I’ll skip the Bogoliubov details: it is done that way in books. Later in this text, another boson Bogoliubov transformation gives the phonon dispersion in the dilute Bose gas (Lec. 7.2) and the fermion Bogoliubov transformation gives the quasiparticle dispersion in BCS theory (Lec. 7.3).

In any case, I think the customary Holstein-Primakoff set-up is needlessly complicated. It’s analogous to setting up phonons by first representing the particle momenta \( p_i \) and deviations \( u_i \) in terms of the creation operators for a set of Einstein oscillators. A Bogoliubov transformation on these would be necessary to obtain the creation operators for the true normal modes.

### 5.5 C Holstein-Primakoff via spin-deviation approach

The starting point is to derive linearized operators that are exactly conjugate. We had \([s_{iz}, s_{iz}] = 2s_{iz}\). Furthermore, for large \( S \), we have \( s_{iz} \approx S \). So, we write

\[
\begin{align*}
    s_{ix} &\equiv \sigma_{ix} + \ldots \quad (5.5.16) \\
    s_{iy} &\equiv \sigma_{iy} + \ldots \quad (5.5.17)
\end{align*}
\]

and

\[
    s_{iz} = \eta_i \sqrt{S(S+1) - s_{ix}^2 - s_{iy}^2} = \eta_i \left[ S + \frac{1}{2} - \frac{1}{2S} \left( \sigma_{ix}^2 + \sigma_{iy}^2 \right) \right] + \ldots \quad (5.5.18)
\]

with commutator

\[
    [\sigma_{ix}, \sigma_{iy}] = i\eta_i S. \quad (5.5.19)
\]

When we plug this in to the Hamiltonian, we get \( \mathcal{H} = \mathcal{H}_{cl} + \mathcal{H}_{sw} + \ldots \), where \( \mathcal{H}_{cl} \) is the classical part (as in Lec. 5.3) and \( \mathcal{H}_{sw} \) is the lowest-order spin wave term. The higher powers would give anharmonicities – interactions between the spin waves. We get

\[
\delta \mathcal{H} = -\frac{1}{2} \sum_{ij} \left[ \sigma_i \cdot \sigma_j - \frac{1}{2} \eta_i \eta_j (|\sigma_i|^2 + |\sigma_j|^2) \right] \quad (5.5.20)
\]

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\(^6\)This is because the operators are keeping track of the deviation of the staggered magnetization from the Néel state; that’s not the same as the deviation of total spin, which is conserved by an exchange Hamiltonian.

\(^7\)E.g. L. R. Walker’s article on spin waves in Rado and Suhl, vol. I.
5.5 D. EQUILIBRIUM FLUCTUATIONS

The term in parentheses came from the \( s_i s_j \) term; we can simplify this term. Namely, \( \sum_{ij} J_{ij} \eta_j \) just \( h^{(0)}/S \), where \( h^{(0)}_i \) is the local field on site \( i \). (It’s commonly the same value on all sites.)

**Fourier transform**

We define the vector \( \mathbf{\sigma}(q) \) by

\[
\mathbf{\sigma}(q) = \frac{1}{N} \sum_{r} e^{i q \cdot r} \mathbf{\sigma}(r) \tag{5.5.21}
\]

Note the offset by \( Q \) in the commutator:

\[
[\sigma_x(q), \sigma_y(q + Q)] = i S \tag{5.5.22}
\]

with all others being zero.

Furthermore, since \( \eta_i \eta_j = \exp(i Q \cdot [r_j - r_i]) \), we have

\[
\sum_{j} J_{ij} \eta_i \eta_j = \tilde{J}(Q). \tag{5.5.23}
\]

Meanwhile, the \( xy \) terms give \( \tilde{J}(q) |\mathbf{\sigma}(q)|^2 \). So we get

\[
\delta \mathcal{H} = \frac{1}{2} \sum_{q} \tilde{K}(q) (|\sigma_x(q)|^2 + |\sigma_y(q)|^2) \tag{5.5.24}
\]

Now we’re set to write the creation/annihilation operators in the usual fashion. In view of (5.5.22), we must pair one direction at \( q \) with the other at \( q + Q \):

\[
b^\dagger(q) = \frac{1}{\sqrt{2S}} (\sigma_x(q) - i \sigma_y(q)) \tag{5.5.25}
\]

\[
b(q) = \frac{1}{\sqrt{2S}} (\sigma_x(q) + i \sigma_y(q)) \tag{5.5.26}
\]

so

\[
[b_q, b^\dagger_{q'}] = \delta_{q=q'}. \tag{5.5.27}
\]

Also

\[
|\sigma_x(q)|^2 + |\sigma_y(q + Q)|^2 = S(b^\dagger_q b_q + \frac{1}{2}) \tag{5.5.28}
\]

hence

\[
\delta \mathcal{H} = \sum_{q} \hbar \omega(q) (b^\dagger_q b_q + \frac{1}{2}) \tag{5.5.29}
\]

with \( \omega(q) \) given by (5.5.7).

5.5 D Equilibrium fluctuations

Having found the dispersion relation and creation operators for magnons, we can plug in to compute the mean-square amplitude of spin deviations due to each mode’s harmonic oscillator zero-point motion (at \( T = 0 \)) or by invoking equipartition of a classical oscillator (at \( T > 0 \)).
One particularly interesting application is to find the lower critical dimension \(d_l\), below which the fluctuations are so big as to destroy long-range order. The dangerous degrees of freedom are the long-wavelength modes, which (due to the continuous symmetry) have a vanishing restoring force. It turns out that for \(T > 0\), \(d_l = 2\); the nonexistence of order in \(d = 2\) is called the “Mermin-Wagner theorem.” At \(T = 0\), the ferromagnet has perfect order in any dimension since the ferromagnetically ordered state is an eigenstate of the quantum Hamiltonian. That isn’t so for antiferromagnets, and it turns out they have \(d_l = 1\) at zero temperature.

A one-dimensional antiferromagnet, then, is necessarily disordered. But it will turn out (see Lec. 5.6) there is more than one way it might be disordered. Quantum spin chains constitute a rich topic—the main cases in which exotic properties (such as fractionalized excitations) are well understood.

**Reduction of ordered moment due to zero-point fluctuations**

The moment reduction is parametrized by \(m_0\) where \(\eta_i m_0 \equiv \langle s_{iz} \rangle_0\). (Recall (5.5.3).) Now, as we used once before, the 1\(S\) expansion gives

\[
\delta m \approx \frac{1}{2S} \langle |\sigma| \rangle = \frac{1}{2NS} \sum_q \langle |\sigma(q)| \rangle,
\]

(5.5.31)

There’s a simple recipe to evaluate \(\langle |\sigma(q)| \rangle\) (This was explained in the phonon context in Modern Models, Lec. 1.5.) The zero-point energy for mode \(q\) is

\[
\frac{1}{2} \langle \hat{K}(q + Q)\sigma_x(q + Q)\sigma_x(q)\rangle + \hat{K}(q)\sigma_y(q)\sigma_y(q)\rangle = \frac{1}{2} h\omega(q)
\]

(5.5.32)

By the virial theorem, each term’s expectation is half the total. Recalling the formula for \(\omega(q)\), we obtain

\[
\langle |\sigma_x(q)| \rangle = \frac{1}{2} \sqrt{\frac{K(q)}{K(q + Q)}}
\]

(5.5.33)

\[
\langle |\sigma_y(q + Q)| \rangle = \frac{1}{2} \sqrt{\frac{K(q + Q)}{K(q)}}
\]

(5.5.34)

With a change of variables \(q \rightarrow q + Q\), we see \(\langle |\sigma_y(q)| \rangle = \langle |\sigma_x(q)| \rangle\), as we expect from rotational symmetry around the \(\hat{z}\) axis.

For an antiferromagnet, then, zero-point fluctuations for a given \(\sigma(q)\) scale as

\[
\langle |\sigma(q)| \rangle \sim 1/|q|.
\]

(5.5.35)

In view of (5.5.31), fluctuations are divergent in the ground state iff the integral of \(1/|q|\) diverges, i.e. \(d_l = 1\) for antiferromagnets as was claimed earlier. \(^8\)

\(^8\)I use “|\sigma(q)|” as shorthand for \(\sigma_x(q)^2 + \sigma_y(q)^2\), and not \(\sigma_x(q)\sigma_y(q)\), which would be different since the operators don’t commute.

\(^9\)We needn’t worry about divergences at large \(q\), since the Brillouin zone is finite. That’s equivalent to saying the inverse lattice constant defines a natural cutoff at large wavevectors.
5.5 X. **SPIN WAVES VIA “CLASSICAL” DYNAMICS**

**Thermal behavior**

The magnon modes are populated according to the Bose-Einstein distribution. Notice that the \(|q|^2\) dispersion of ferromagnetic magnons leads to different behavior than the same dispersion of free boson atoms. The key difference is that magnons have zero chemical potential, since magnon number is not conserved, whereas the number of atoms is fixed corresponding to nonzero chemical potential.

It’s easy to work out that the low-\(T\) specific heat is \(C(T) \sim T^{d/2}\) (ferromagnet) or \(T^d\) (antiferromagnet), in \(d\) spatial dimensions. (The latter form is, of course, the same as that of phonon specific heat; this form only depended on the linear dispersion at low energies and the boson statistics.)

The long wavelength properties of any quantum phase involve many degrees of freedom and are hence invariably *classical*, even when that phase owes its very existence to quantum mechanics (e.g. a superfluid). Indeed, at sufficiently long wavelengths, a Goldstone mode has \(\omega(q) \rightarrow 0\) so eventually \(\hbar \omega \ll T\) and we replace the Bose-Einstein by the Boltzmann distribution. The result is equivalent to the statistical mechanics of purely classical spins (i.e. where \(s_i\) is just a c-number vector of fixed length). This regime is valid when \(T \ll JS^2\) (where \(J\) means a typical exchange interaction) but \(T \gg \hbar \omega\) (where \(\hbar \omega\) means a typical spinwave energy, analogous to the Debye energy). Since \(\hbar \omega \sim JS\), strictly speaking this regime can be valid only when \(S \gg 1\).

You can go further to work out the moment reduction of the classical ferromagnet and antiferromagnet in this regime. In this case,

\[
\langle |\sigma(q)|^2 \rangle \sim \frac{T}{K(q)}
\]

— just a form of equipartition. Hence \(\delta m \sim \int d^d q (1/|q|^2)\), implying a critical dimension \(d_l = 2\) as claimed above. The same was true for phonons: thermal fluctuations always destroy order for \(d \leq 2\).

5.5 X **Spin waves via “classical” dynamics**

Here is an alternate (even more intuitive) path to spin waves.\(^{10}\) We start from (5.5.1) and treat the spins as c-numbers of fixed length \(S\), obeying the semiclassical spin dynamics derived in *Modern Models* Lec. 5.0 C.

\[
\frac{ds_i}{dt} = \frac{1}{\hbar} s_i \times h_i
\]

Eq. (5.5.37) is the analog of Newton’s equations for particles.

The “local field” \(h_i\) is the sum of the exchange interactions of all spins interacting with \(s_i\):

\[
h_i = \frac{\delta H}{\delta s_i} = \sum_j J_{ij}s_j
\]

This (or at least its component normal to \(s_i\)) is analogous to a force acting on \(s_i\). However, like a charged particle in a very large field, and just like a gyroscope, the spin moves perpendicular to this force. Indeed, that must be the case in order to conserve the energy; unlike the potential energy in a mechanics problem, here \(H(\beta_i)\) is

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\(^{10}\)The convention for numbering sections in these lectures from “Modern Models” is that A,B, ..., are necessary, while X, Y, ... are appendices, examples, or advanced supplements.
the complete Hamiltonian: there is no separate kinetic energy term. In a spin problem, one component of a spin is canonically conjugate to the other component; thus, terms of the same kind in $\mathcal{H}$ play the role of both kinetic and potential energy terms.

Note the following properties of the dynamics (5.5.37):

(i) preserves each spin’s fixed length;
(ii) conserves total spin
(iii) conserves total energy.

Linearizing the equations of motion

The game is to start with a classical ground state of (5.5.1), as discussed in Modern Models Lec. 5.3, and linearize the equations (5.5.37) for small deviations around this state. Let $s_i$ be the vector of small deviations

$$s_i = s_i^{(0)} + \sigma_i.$$ (5.5.39)

It can be shown (see (Ex. 5.5.1)) that the linearized equations of motion are

$$\frac{d}{dt}\sigma_i = \frac{1}{\hbar} s_i^{(0)} \times [-h_i^{(0)} \sigma_i + S \sum_j J_{ij} \sigma_j]$$ (5.5.40)

Here $\sigma^{(0)} = Sn_i^{(0)}$; also, $h_i^{(0)} = |h_i^{(0)}|$ with $h_i^{(0)} \equiv \sum_j J_{ij} s_j^{(0)}$.

Notice that if you apply an infinitesimal rotation uniformly to all the spins in the ground state, i.e. $\sigma_i = \delta \theta \times s_i^{(0)}$, the quantity “[...]]” in (5.5.40) is zero hence $d\sigma/dt = 0$. This is the $q = 0$ Goldstone mode. (Its analog for a solid: all forces and accelerations would be zero if we applied the same infinitesimal translation to all atoms.)

The next step is to Fourier transform the linearized equations. Translational symmetry guarantees that the result is diagonal in $q$-space: that is, the time derivative of a deviation at wavevector $q$ is a linear combination of deviations at the same $q$. However, there are $2m$ independent deviation modes at each wavevector, where the $2$ counts the two directions transverse to each spin, and $m$ counts the number of sites (basis) in the magnetic unit cell: $m = 1$ for a simple ferromagnet, $m = 2$ for a simple antiferromagnet. Just the same was true for phonons: that is, if the Bravais unit cell had a basis of $m$ atoms, the number of modes was $dm$, where $d$ was the number of components of infinitesimal displacements.) Consequently, the frequencies $\omega(q)$ (times $i$) are eigenvalues of a $2m \times 2m$ matrix.

For a simple (collinear) ferromagnet or antiferromagnet, $s_i^{(0)}$ is along the $\pm \hat{z}$ axis for every site. Let

$$(\sigma_{ix}, \sigma_{iy}) = \frac{1}{\sqrt{N}} (\tilde{\sigma}_{qx}, \tilde{\sigma}_{qy}) e^{iqr},$$ (5.5.41)

Then, $x$ components inside the square brackets in (5.5.40) cause only $y$ components of $d\sigma/dt$, and vice versa, just like the position and momentum of the harmonic chain. In going to Fourier space, one must, however, be careful about the wavevector. If $s_i^{(0)} \propto e^{iQr}$, with an ordering wavevector $Q$, then inserting a dependence $e^{iqr}$ in the right hand side, we get a dependence $e^{i(q+Q)r}$ on the left hand side.\footnote{The coupling of $\sigma(q)$ with $\sigma(q+Q)$ doesn’t contradict the earlier statement that the dynamics is diagonal in Fourier space. For the magnetic unit cell, $Q$ is a reciprocal lattice vector, since a translation $R$ leaves the ground-state spin configuration unchanged iff $e^{iQR} = 1$. Thus, the Brillouin zone for the magnetic lattice gets folded smaller by the same factor (2 in this case) as the magnetic cell got enlarged (in real space) from the unit cell of the spin model itself. In particular, $\sigma(q+Q)$ is a second mode at a wavevector equivalent to $q$ (in the magnetic zone).}
With the caution just mentioned, the end result for the usual two-sublattice antiferromagnet is a $2m \times 2m$ set of linear equations, for each $q$. Here $m = 2$ and the explicit equations are:

$$\frac{d}{dt} \sigma_x(q) = \tilde{K}(q)\sigma_y(q + Q) \quad (5.5.42)$$

$$\frac{d}{dt} \sigma_y(q) = -\tilde{K}(q)\sigma_x(q + Q). \quad (5.5.43)$$

Finally we obtain $d^2\sigma(q)/dt^2 = -\omega(q)^2\sigma(q)$, implying the dispersion $\omega(q)$ as advertised. The magnons are obtained by quantizing these harmonic oscillators.

**Polarizations**

In the AFM case, with simple Néel order, it is possible to make circularly polarized modes – linear combinations, with relative phase $\pm i$, of the two polarizations just described. In a circular polarized mode, the spin deviations just precess around each $s_i^{(0)}$ at frequency $\omega$. It can be shown that the spin wave mode, when quantized, carries a net $\pm$ spin (summed over the entire lattice) of exactly $+1$ or $-1$, depending on the sense of circular polarization, where $z$ is the staggered magnetization axis. That can only be possible if the deviations are different on the even and odd sublattices: for example, in order to create a net change of spin $-1$, the transverse deviations on the up sublattice must be large, and on the down sublattice must be small.

**5.5 Y  Goldstone mode behaviors**

The reason a ferromagnet has a quadratic dispersion (5.5.10) but the antiferromagnet has a linear dispersion (5.5.12) is that the conserved quantity (spin) is the same as the order parameter for the ferromagnet, but not for the antiferromagnet (its order parameter is the staggered spin).

The Goldstone coordinate is always a long wavelength twist of the spin configuration. (Here “twist” means a rotation, by an angle or rotation axis which is slowly varying in space.) That twist is just the deviation at wavevector $q + Q$. By symmetry, rotation by a uniform angle would cost nothing, so the energy cost of the long-wavelength twist must be proportional to $|\nabla s|^2 \sim |q|^2$; hence the restoring force for that mode scales proportional to $|q|^2$. The above argument is good in any system coupled by exchange (so that it has a rotational symmetry).

The variable conjugate to a long-wavelength twist is always a long-wavelength modulation of the magnetization itself. So in the oscillating dynamics, a twist makes a magnetization and vice versa.

Meanwhile the restoring force for a uniform magnetization is simply the inverse susceptibility, which is divergent only in a ferromagnet.

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12 The other two of the $2m = 4$ equations promised are merely obtained by shifting $q \rightarrow q + Q$ in the above equations; so, we have just two equations, valid throughout the original (not just the reduced) Brillouin zone.


14 If we added an external magnetic field, we would get precession around that field. If you added a field and let the spins develop a magnetization in response to it, you would have a classical ground state – each spin aligned with its local field – and no precession. And the net response is linear in the field and spin deviations. Thus, if you have no field but do have a magnetization, the spins must precess around that axis.
Anisotropy

What if the Hamiltonian includes terms that violate spin-rotation symmetry? An example would be an easy-z-axis term such as the single-site anisotropy term \( \frac{1}{2} D_0 (s_{ix}^2 + s_{iy}^2) \). We can linearize this Hamiltonian too, and the consequence is just the modification \( \tilde{J}(q) \rightarrow \tilde{J}(q) + D_0 \).\(^{15}\)

Thus in the ferromagnet,

\[
\omega(q) \rightarrow \omega(q) + SD_0 \propto (\text{const} + |q|^2) \quad (5.5.44)
\]

For small-\(q\) modes in the antiferromagnet, provided \( D_0 \) is smaller than the exchange constants, the addition hardly changes the large coefficient for the magnetization deviation, \( \tilde{K}(q+Q) \). But the addition dominates the coefficient for the Goldstone deviation since (without it) \( \tilde{K}(q) \rightarrow 0 \) as \( q \rightarrow 0 \). Thus

\[
\omega(q) \propto [(O(J) + D_0)(O(J)aq^2) + D_0]^{1/2} \approx \hbar \sqrt{\omega_0^2 + c^2|q|^2} \quad (5.5.45)
\]

In Sec. 5.5 D, anisotropy will provide a cut off for the long-wavelength divergences that could occur in low dimensions. Algebraically, \( \tilde{K}(q) \) [Eq. (5.5.5)] acquires a constant offset of \( D_0 \) (say) while the second term does not; hence, as \( q \rightarrow 0 \), \( \tilde{K}(q) \rightarrow \text{constant} \).

A corollary is that a one-dimensional spin chain can remain ordered at \( T = 0 \), if it has an Ising-like anisotropy. (But it will disordered as soon as \( T > 0 \), for the same reason a classical Ising model is disordered.)

**Electron spin resonance and anisotropy**

Besides neutron diffraction, another experiment that probes spin dynamics is *electron spin resonance* done at microwave frequencies. This essentially couples to the \( q = 0 \) spin wave mode. An ideal Goldstone mode, of course, this mode has zero frequency at \( q = 0 \), but the anisotropy gap brings it up to finite frequency.

**Exercises**

**Ex. 5.5.1 Linearized equations of motion**

Derive (5.5.40) and (??).

OPTIONAL. Verify the properties (i)-(iii) mentioned after (5.5.40).

\(^{15}\)Just as \( \tilde{J}(q) \) was analogous to an effective spring constant \( \tilde{K}(q) \) of the phonon case, so \( D_0 \) is analogous to the Einstein term that was added to it.
5.5 Y. GOLDSTONE MODE BEHAVIORS

Ex. 5.5.2 1D Ferromagnet

You are given a chain with ferromagnetic $J$ acting between nearest neighbors only. Derive the spin-wave dispersion relation (not assuming (5.5.9) and find the coefficient $D$ in (5.5.10).

Ex. 5.5.3 1D Antiferromagnet – frequencies

Consider a 1D antiferromagnetic chain, with nearest neighbor exchange $-|J|$. Find the linearized and Fourier transformed equations of motion. You ought to find $\sigma_x(q)$ and $\sigma_y(q + \pi/a)$ coupled by first order equations, similarly for $\sigma_x \leftrightarrow \sigma_y$.

What are the frequencies $\omega(q)$?

At what point(s) in the Brillouin zone does $\omega(q)$ go to zero?

Which Brillouin zone would you use for labeling modes? How many polarizations are there at each $q$? Is the total number of modes the same as in the ferromagnetic case?

Note: Really, the ground state of a 1D antiferromagnet never has ordinary Néel order; it either decays algebraically or exponentially, depending whether $S$ is half-odd or integer valued. (See Lec. [5.6]). However, when $S$ is large, the order decays so slowly that it might as well be long range ordered.

Ex. 5.5.4 1D ferromagnet and antiferromagnet – modes

(a). Ferromagnet – Look at A&M Fig. 33.7(b) for the ferromagnet. Sketch the pattern of the spin deviations at a time 1/4 period $(\pi/2\omega)$ later.

(b). Antiferromagnet –

Consider a linearly polarized mode of a lattice where the staggered magnetization is along $\pm \hat{z}$. At time 0 the deviations are ferromagnetic, (say) all spins towards $+x$.

Draw a sketch of an antiferromagnetic mode at time 0 and also at a moment 1/4 period later. At which of these moments are the deviations a rotation of the spins by a slowly varying angle? What would be the analogs of the above two snapshots, for a classical lattice normal vibration mode?