5.5 Z “Ordering due to disorder”

When one identifies the best modes of the $J_{ij}$’s by the Luttinger-Tisza (Fourier) method of Lec. 5.3, if there is frustration it can easily happen that there is are more than two points $Q$ in the Brillouin zone at which $J(Q)$ is maximized. (In some “highly frustrated” cases – the kagomé lattice of Fig. 5.5.1(b), or its 3D analog the pyrochlore lattices – the maximum is obtained at every $q$ in the zone!) In such a case, there may be a variety of ways to associate spatial modes with components of the spin vectors: then, there is a continuous family of ground states which have degenerate classical energies, parametrized by a set of real numbers, say $\{f_1, f_2, \ldots\}$. Of course, rotational symmetry generates a continuous family of ground states, but here we have in mind states that are not symmetry related.

Since the different states are not symmetry-related, they (usually) don’t have the same spectrum of spin-wave (= magnon) excitations. Let’s consider the spin-wave zero point energy, summed over all modes and implicitly depending on the parameters that label which state we’re fluctuating around:

$$E_{sw}(\{\Phi_j\}) \equiv \frac{1}{2} \hbar \sum_k \omega_k$$

(5.5.1)

(here $k$ includes wavevector, polarization, and band indices). Notice $E_{sw}$ is next order term after the classical one $E_{cl}$ in the systematic (Holstein-Primakoff) expansion in $1/S$ of the ground state energy; we can also view it as a sort of effective Hamiltonian for the coordinates $\{\Phi_j\}$.

Then $E_{sw}$ (usually) splits the classical degeneracy, selecting a particular parameter set $\{\Phi_j\}$ (and a discrete list of others, related to it by symmetries). Because an ordered state is determined by the disordered fluctuations, this has been dubbed “order due to disorder” (This name is especially appropriate in situations where the degeneracy causes a loss of long-range order, if the spin-wave zero point term is neglected.)

The $J_1$-$J_2$ antiferromagnet is perhaps the simplest example: it has one parameter $\Phi$ which is the angle between the spin axis of the even sites and that of the odd sites. Thus, the family of states run from the *collinear* – meaning all spins are parallel or antiparallel to the same direction – to the opposite kind of states, in which the spins are as un-parallel as can be. Some very general arguments show that the (harmonic) spin-wave energy always favors the *collinear* states. \(^1\) See (Ex. 5.5.5).

Exercises

Ex. 5.5.5 “Order by disorder” in the $J_1$-$J_2$ square lattice

Take the 2D square lattice with antiferromagnetic first and second neighbor couplings $-J_1$ and $-J_2$ (so $J_1$ and $J_2$ are positive numbers). Consider the case $J_1 = 0$: the even and odd sublattices are decoupled, each having a simple order with directions $\pm \Phi / 2$ or $\pi \pm \Phi / 2$, where $\Phi$ is the angle between spin directions. The special case $\Phi = 0$ ($\Phi = \pi$ is equivalent by a symmetry) is collinear, in that all spins are parallel (or antiparallel) to the same axis in spin space.

(a). This degeneracy, it turns out, is retained when $J_1/J_2$ is nonzero but not too large. Show that if we assume a $(\pi, 0)$ kind of state (shown in Fig. 5.5.1(a)), the energy is independent of $J_1$. Compute the classical energy/spin for this state, and compare it to that of the usual $(\pi, \pi)$ order with the same interactions; what is the critical value of the ratio $J_1/J_2$ at which the $(\pi, 0)$ state wins?

Also verify that the local field $h(0)$ is the same on every site.

(b). The above isn’t enough to say these are the stable states: why couldn’t the classical ground state be, e.g., an incommensurate spiral, with its ordering wavevector $Q$ smoothly passing from $(\pi, 0)$ to $(\pi, \pi)$ as $J_1/J_2$ is varied? That can be pinned down using the Luttinger-Tisza method of Lec. 5.3.

Write the formula for $J(q)$ as $J_1 C_1(q) + J_2 C_2(q)$. [Recommended: use this to do Ex. 5.3.3(a), or part of it.]

(c). Set up the calculation for $\omega(q; \Phi)$ for any one of these ground states. This is most simply done, I believe, starting from the classical equations of motion (5.5.40). In the present case, all the classical spin directions point in the $xy$ plane; let $\phi_i$ be their angle on each site. Define in-plane deviations of that angle as $\sigma^i$; the out of plane deviations are just $z_i$, which I’ll write $z_i$ for short. Verify that the equations are

$$\frac{dz_i}{dt} = h(0) \sigma_i - J_{ij} \cos(\phi_i - \phi_j) \theta_j$$

$$\frac{d\phi_i}{dt} = h(0) z_i - J_{ij} z_j.$$  \hspace{1cm} (5.5.2)

Notice that, although the magnetic unit cell is $4 \times 4$, the unit cell of these equations is just $1 \times 1$. (That was a benefit of using the deviations in terms of $\theta_i$ rather than worry about the actual directions of the spin deviations.)

(d). Write the Fourier version of these equations in terms of $J_1$, $J_2$, $C_1(q)$, and $C_2(q)$. Notice the only dependence on $\Phi$ is that, in one of the two equations, the $J_1$ term looks like $J_1[\cos(\Phi) D_1(q)]$.

(e). Write the integral over the Brillouin zone which represents $E_{\text{sw}}(\Phi)/N$, the zero-point energy per spin.

Consider the case that $J_1$ is a small parameter, and Taylor expand the square root containing $J_1 \cos \Phi$ to second order. (Recommended: first write the other term with a compact notation, maybe $\tilde{K}_2(q) \equiv J_2[\tilde{C}_2(q) - \tilde{C}_2(q)]$.) Why does the first order term cancel? Your final result should have the form

$$E = \text{const} + B \cos^2 \Phi$$

where $B$ is expressed as an integral. Don’t try and do this integral.

(optional) Explain why the integrand isn’t actually divergent, even though it should have a denominator that vanishes at certain points in the zone.