Lec. 14: Momentum-space renormalization group

A. Overview of the $\epsilon$ expansion

This is a perturbative way to obtain exponents as a power series in $\epsilon$, where $d = 4 - \epsilon$.

Advantages and disadvantages

+ Systematic
+ Can treat all $n, d$ at once. (Where $n$ is number of spin components in the $n$-vector spin models, $d$ is spatial dimension).

Disadvantages

- Must start from field theory (LGW Hamiltonian).
  (Thus hard to connect it quantitatively to microscopics. Good for exponents at the critical point, but mostly useless for the global phase diagram.)
- Only good near $d_u = 4$ (bad for $d = 2$).
  [Furthermore, the series in $\epsilon \equiv 4 - d$ are asymptotic; indeed, they already get worse after $O(\epsilon^2)$. To make the higher orders useful, sophisticated resummations must be performed.]

Two routes to the $\epsilon$ expansion.

We’ll do the more elementary “momentum-shell” approach. The alternative field-theoretical approach is set up similar to quantum field theory.

Both methods give the same answers. The momentum-shell way is easy to $O(\epsilon)$, doable to $O(\epsilon^2)$, but for higher orders use the field-theoretical approach. Goldenfeld (Chapter 12) presents the momentum-shell way, but (in intro to Ch. 12) sketches the field-theoretical way; for a lengthy, pedagogical intro to the latter, see The theory of critical phenomena by Binney, Dowrick, Fisher, and Newman.

Results of $\epsilon$ expansion.

Let $d = 4 - \epsilon$. Then:

$$\gamma = 1 + \frac{(n + 2)}{2(n + 8)}\epsilon + \frac{(n + 2)}{4(n + 8)^2}(n^2 + 22n + 52)\epsilon^2 + \ldots$$  \hspace{1cm} (0a)

$$\eta = 0 + \frac{(n + 2)}{2(n + 8)}\epsilon^2 + \ldots$$  \hspace{1cm} (0b)

Hyperscaling is satisfied, so we can find all the other exponents from these two.

References: Goldenfeld (1992), Chap. 12, and (for details) Ma’s Modern Theory of Critical Phenomena, (1976). For these notes, I also used my lecture notes from a course given at M.I.T. by Amnon Aharony (1980). I use the old traditional momentum-space version of the cutoff, but have simplified my presentation by writing it (mostly) in real space.
Overview of the path

As I’ve mentioned, there are basically two aspects of any RG calculation:

- computing the iteration (harder part), and
- extracting the exponents from the flows.

This writeup is all about the first aspect. That, in turn, typically breaks into two parts:

(i) a partial summation of the partition function over some degrees of freedom, obtaining an effective Hamiltonian in the remaining variables which I’ll call $H_{\text{sum}}$.

(ii) putting this “summed” Hamiltonian back into the original functional form (but with new parameter values), which I’ll call $H_{\text{new}}$.

[Note: subscript “$\ldots$\text{new}” will generally denote “new value of parameter after completing one R.G. iteration”; in prior lecture notes the notation was “$\ldots$\text{old}”.]

Since the story is long anyhow, I’ve elected to postpone as much commentary and qualifications as possible, to a supplementary section. Be watching out for possible terms that seem to be ignored: see if you can guess why:

1. zero by symmetry;
2. zero by disjointness of wavevectors in the “momentum-shell” $|\mathbf{q}| \in [\Lambda, \Lambda(1 + \delta \ell)]$ (see below) from those in the sphere $|\mathbf{q}| < \Lambda$;
3. zero by smallness of the parameters $t$, $u$, or $\delta \ell$
4. irrelevance (in RG parlance) of the term.

B. Set-up of Hamiltonian

We use the Landau-Ginzburg-Wilson (LGW) Hamiltonian

$$
H = \int d^d \mathbf{r} \left[ \frac{1}{2} at S^2 + \frac{1}{2} \gamma |\nabla S|^2 + \frac{1}{4} u S^4 \right].
$$

As you recall, $t$ is the temperature variable. * From here down, I’ll adopt a shorthand $\int d^d \mathbf{r} \to \int$.

This is a field theory: the “spin” is a smooth function $S(\mathbf{r})$, which can take on a continuum of values. Many other terms are allowed in (1) by symmetry such as $S^2|\nabla S|^2$ or $S^6$, etc.. These all turn out to be “irrelevant”, at least for $d < 4$ (see Lec. 2.7 F 1, below). Also, I’ve omitted $-h\phi(\mathbf{r})$ for simplicity [hence (1) is even in $S$, a key symmetry.] You must go back and put it in, to obtain the critical exponent associated with magnetization.

0. For the big picture: an important point

We’ll be expanding in powers of (some) non-gradient terms. This will be systematically meaningful because the coefficients $t$ and $u$ are small parameters. Certainly we expect $t$ small near the critical point – in Landau theory $t_c$ is exactly zero (it’s changed a little here where we take account of fluctuations). But how come $u$ is small – that certainly

* I keep the notation from Lec. 2.3– remember “$a$” in “$at$” is not the lattice constant! Goldenfeld and others set $a \equiv 1$. 
isn’t generically true! Well, it turns out that the fixed point $u^*$ is $O(\epsilon)$ where $d = 4 - \epsilon$; that’s the entire reason for working in “3.99” dimensions. Thus at the end, we’ll obtain RG eigenvalues (and implicitly exponent values) which depend on $u^*$ as a power series; then expressing $u^*$ as a function of $\epsilon$, that power series gets converted into the $\epsilon$ expansion I quoted in eqs. (0).

For now, we set up with general $d$; only later on do we need to specialize to $\epsilon \ll 1$ (where $d \equiv 4 - \epsilon$)

1. Cutoff $\Lambda$

We must implement a shortest length scale for fluctuations in $S(\mathbf{r})$ analogous to a lattice constant $a$. So we mock up the effect of a lattice as follows: instead of integrating over a Brillouin zone in reciprocal space, we integrate over a sphere

$$|\mathbf{q}| < \Lambda.$$ 

In other words, we’re allowing all functions in real space that are built out of wavevectors $< \Lambda$. Obviously $\Lambda \sim 1/a$ where $a$ is roughly the spacing between degrees of freedom, the lattice constant. All these Fourier modes vary on a scale of $\sim a$ or slower, so that $S(\mathbf{r})$ varies on the scale of $a$. Thus

$$S(\mathbf{r}) \equiv \frac{1}{\sqrt{\Omega}} \sum_{|\mathbf{q}| < \Lambda} e^{i\mathbf{q} \cdot \mathbf{r}} S_{\mathbf{q}}$$

(2)

Note the convention for the Fourier transform: to conveniently count degrees of freedom and write integrals over $\{S(\mathbf{r})\}$, either in real space or in “momentum” space, we’ve imagined having a finite system, so the sum over $\mathbf{q}$ is discrete. The conversion formula is $\Omega = N\nu_{\text{cell}}$, where $\Omega$ is the $d$-dimensional “volume” of the system.

The partition function is

$$Z = \prod_{|\mathbf{q}| < \Lambda} \left( \int dS_{\mathbf{q}} \right) e^{-\mathcal{H}}$$

(3)

2. Functional R.G. notion

My personal choice is to view the R.G. from a somewhat unusual slant. Write

$$\mathcal{H} = \int \left[ \frac{1}{2} \gamma |\nabla S(\mathbf{r})|^2 + U(S) \right]$$

(0’)

with a *general* Landau term $U(S)$. The slant is that renormalization gives us a new function $U_{\text{new}}(S)$ with a slightly different shape. To anticipate, certain of the new contributions to $U(S)$ make the wells in (1’) deeper and the barrier higher. That corresponds to a more ordered state, as if we decreased $t$, so we could call these “energy” effects. Other terms tend to wash out the double-well (so it looks more like ‘a single-well); That is, they push
$U(S)$ towards a shape that describes a more disordered state, as if we increased $T$, so call these “entropy” effects. A critical fixed point is when $U_{\text{new}}(S) = U(S)$, which implies the energy and entropy effects just balanced.

This kind of iteration – when, given the whole function $U(S)$, you construct a whole new function $U_{\text{new}}(S)$ – is called a “functional renormalization group” Here $a$ and $u$ are simply convenient for parametrizing the function’s shape (and for extracting the RG eigenvalues).

The basic notions don’t really depend on writing $U(S)$ as a polynomial. Only at the end will we specialize to the Landau-Ginzburg-Wilson (LGW) form

$$U(S) = \frac{1}{2}a S^2 + \frac{1}{4}u S^4$$

(1')

For brevity I’ll write derivatives

$$U'(S) \equiv \frac{dU(S)}{dS}; \quad U''(S) \equiv \frac{d^2U(S)}{dS^2}.$$

In fact, emphasizing $U(S)$ and writing in real space is good for handling the expansion in the anharmonic terms; on the other hand, the more traditional approach of working in momentum space is good for handling the quadratic terms and doing the integrals.

C. Setup of Renormalization group

There are two pieces of the set-up defining how the degrees of freedom are eliminated (“spin map”) and how the Hamiltonian is broken up.

1. Spin map and rescale factor:

The procedure is that we “decimate in Fourier space.” That is, we resolve

$$S(r) \equiv S_< (r) + \sigma(r)$$

(4)

where $S_< (r)$ (read it “$S$ lesser”) contains only Fourier modes with $q < \Lambda/b$ and $\sigma(r)$ includes the remaining ones in the shell $\Lambda/b < q < \Lambda$.

Clearly $N_{\text{new}} = N/b^d$ (number of Fourier modes in integral), so we indeed reduce the number of degrees of freedom in the usual fashion.

We let

$$b = 1 + \delta l$$

(5)

where

$$\delta l \ll 1$$

(5')

Eq. (5') ensures that $\sigma(r)$ is small, since only a thin shell of Fourier space contributes. Later in the calculation, we need only keep terms that will be $O(\delta l)$. So, you see that (with $t^*$ and $u^*$), we have two independently small parameters in this approach.
New states – will just be the configurations of $S_<$. Those configurations go with $H_{\text{sum}}$, so they aren’t quite the same as the final $S_{\text{new}}(r)$. For that, we’ll need a final rescaling of the cutoff (Sec. 2.7 F, below)

Sketch of decomposition (4):

2. Definition of $H_0$:

The spin projection rule (4) naturally implies a 3-way splitting of the Hamiltonian into terms $H_<$ that only involve $S_< (r)$, terms $H_\sigma$ that only involve $\sigma (r)$, and terms $\hat{V}$ that couple $S_< (r)$ to $\sigma (r)$. This is simplified by the fact that $\sigma$ is small – recall we found $\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2) \rangle_0 = O(\delta l)$. All cross terms come from non-quadratic terms in the Landau free energy, which we just Taylor expand:

$$U(S) = U(S_< + \sigma) = U(S_<) + U'(S_<)\sigma + \frac{1}{2}U''(S_<)\sigma^2 + ...$$  \hspace{1cm} (6)

For the $O(\epsilon)$ R.G., we won’t need $O(\sigma^3)$ or higher powers in (6). Since we eventually want $H_{\text{sum}}$ contributions of order $\delta l$, i.e. $O(\sigma \sigma)$, we’ll need $\hat{V}$ up to $O(\sigma \sigma)$ for the $O(\hat{V})$ cumulant, but only to $O(\sigma)$ in the $O(\hat{V}\hat{V})$ cumulant term.

$$H_< = \int [U(S_<) + \frac{1}{2}\gamma|\nabla S_<(\mathbf{r})|^2];$$  \hspace{1cm} (7a)

$$H_\sigma = \int \frac{1}{2}\gamma|\nabla \sigma(\mathbf{r})|^2$$  \hspace{1cm} (7b)

$$\hat{V} = \int [U'(S_<)\sigma + \frac{1}{2}U''(S_<)\sigma^2 + ...]$$  \hspace{1cm} (7c)

In that integration, $H_<$ [eq. (7a)] plays a trivial role: we’re summing a partial partition function conditional on a fixed $S_< (r)$, configuration, so $H_<$ just behave as a constant and passes through undigested to $H_{\text{new}}$

The expectations over the configurations of $\sigma (r)$ are written $\langle \ldots \rangle_0$; they are taken in the ensemble weighted according to $H_\sigma$ in (7b). The whole calculation is tractable because such such expectations are trivial, which depends on two facts:

(i) $H_\sigma$ is quadratic, hence the $\{ \sigma(\mathbf{r}) \}$ are Gaussian random variables;
(ii) **Important!** the coefficients of $\sigma$ must be independent of $S_\prec$.

Since $\nabla S^2 = (\nabla S_\prec + \nabla \sigma)^2$, why doesn’t a cross term $\int \nabla S_\prec \nabla \sigma$ appear in (7c′)? In fact, it vanishes, as does the cross term $t \int S_\prec \sigma$. For

$$\int S_\prec (r) \sigma (r) \rightarrow \sum_q S_\prec (q) \sigma (-q) \equiv 0 \quad (8)$$

since $S_\prec$ and $\sigma$ have no Fourier components in common; i.e. they are orthogonal functions. (A corollary of this fact is that $\sigma (r)$ is very wiggly.)

**[EXERCISE (a): Confirm eq. (8).]**

Similarly for quadratic factors involving gradient terms, we get a similar Fourier sum but with a $q^2$ factor.

3. **Program using cumulant expansion**

*The big picture:* We’re applying ideas of “conditional expectation” to the partition function. Given a configuration $S_\prec (r)$, we want to integrate the partition function over $\sigma (r)$. That integration is doable only because the latter integrals are Gaussian: as $\sigma$ was small, we could throw away terms in $H$ beyond $O(\sigma^2)$.

The cumulant expansion says

$$H_{\text{sum}} = \langle H_{\prec}(\{S_\prec\}) + F_{0}^{>}\rangle + \langle \hat{V} \rangle_0 - \frac{1}{2} [\langle \hat{V}^2 \rangle_0 - \langle \hat{V} \rangle_0^2] + \ldots \quad (9)$$

Here the term $F_{0}^{>} = - \ln Z_0^{>}$ is the free energy you get from doing the integrals over the $\sigma$ components. However since $H_\sigma$ is independent of $\{S_\prec\}$, $Z_0^{>}$ just contributes a constant to the new Hamiltonian, and as in the other R.G. calculations we did (e.g. Lec. 2.6A), we’re not keeping track of it. Computing the first-order and second-order expectations in (9) is the hard part. It turns out that to get exponents to $O(\epsilon)$ we need both the $O(\hat{V})$ and the $O(\hat{V}^2)$ cumulants.

Our program is to stick (7c) into the cumulant expansion (9). Then the goal will be to put this result into the form

$$H_{\text{sum}}(\{S_\prec\}) = \int \frac{1}{2} t_{\text{sum}} S_\prec^2 + \frac{1}{2} \gamma_{\text{sum}} |\nabla S_\prec|^2 + \frac{1}{4} u_{\text{sum}} S_\prec^4. \quad (10)$$

where $t_{\text{sum}}, \gamma_{\text{sum}}, u_{\text{sum}}$ contain contributions from the first and second cumulants. Here the subscript “(...)_sum” indicates that we have gathered together all the statistical weight.

To anticipate, under renormalization, one actually generates many terms besides those in (10), which involve higher powers and/or higher derivatives of $S$. But (it can be shown) these are all “irrelevant” in the R.G. sense: the most basic terms (in Landau’s sense) turn out to be the most important ones here. So let’s keep (10) in mind as a target while we grind through (9).

**D. Gaussian Hamiltonians and evaluating Gaussian expectations**

Compare Cardy Sec. 5.4 and 5.5. I have not written all I would like about the Gaussian model, meaning justgg omit the quartic term.
0. A preview: necessary technology

To evaluate expectations for any set of Gaussian random variables, we use Wick’s theorem, which says that if \( \{ \varphi_i \} \) are Gaussian random variables, then

\[
\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle_0 = \langle \varphi_1 \varphi_2 \rangle_0 \langle \varphi_3 \varphi_4 \rangle_0 + \langle \varphi_1 \varphi_3 \rangle_0 \langle \varphi_2 \varphi_4 \rangle_0 + \langle \varphi_1 \varphi_4 \rangle_0 \langle \varphi_2 \varphi_3 \rangle_0
\]  

(11)

and similarly for products with any number of Gaussian factors: just break it up into pairs in all possible combinations.

Since \( \varphi(\mathbf{r}) \) is just a linear combination (2) of the \( \varphi_{\mathbf{q}} \)’s, it follows that the \( \varphi(\mathbf{r}) \)’s also satisfy Wick. Note that the Wick formula also tells that \( \langle \varphi(\mathbf{r})^4 \rangle_0 = 3 \langle \varphi(\mathbf{r})^2 \rangle_0^2 \), since there are 3 ways of pairing the factors. This is probably well known to you as a basic property of Gaussians; but it’s nice that Wick handles it properly, because that saves us from making special cases in sums over \( \mathbf{r} \) and this simplifies the algebra (compare XY model exercise, HW 3.1).

Gaussian correlation functions in LGW case

Now to specialize to the case at hand. Of course, \( \langle \sigma(\mathbf{r}) \rangle_0 = 0 \) (and all odd powers vanish). The even-power expectations can be reduced to pair expectations by Wick’s theorem. So the building block of expectations is the correlation function (= Greens function) of the \( \sigma \)’s

\[
\langle \sigma(\mathbf{r}) \sigma(\mathbf{r}') \rangle_0 \equiv G_G(\mathbf{r} - \mathbf{r}')
\]

Important: what makes this all work is that, when written in Fourier space, the terms in different \( \mathbf{q} \) are independent:

\[
\mathcal{H}_\sigma \equiv \sum_{\mathbf{q}} (\gamma |\mathbf{q}|^2) |\sigma(\mathbf{q})|^2
\]

Then, by using equipartition and Fourier transforming back to real space, we obtain the two-point functions.

\[
G_G(\mathbf{R}) = \frac{1}{\Omega} \sum_{\mathbf{q}} \frac{e^{i\mathbf{q} \cdot \mathbf{R}}}{\gamma |\mathbf{q}|^2}
\]  

(12)

In (12) I introduced a shorthand notation \( \sum_{\mathbf{q}} = \sum_{4 \pi < |\mathbf{q}| < \Lambda} \) for integrals over the shell.

We already evaluated this sort of expression in Lec. 2.4 (correlations in mean-field theory). There is a technical difference: the denominator in (12) is the same as in Lec. 2.4, but here we are summing only over a narrow range of wavevectors. That actually makes it much easier to evaluate it explicitly, as we have to do at some point (observe that the magnitude \( |\mathbf{q}| \) is practically constant in the domain of integration in (12), and the integrand depends only on \( |\mathbf{q}|^2 \)).

Here are a couple special expectations that appear commonly in the coefficients:

\[
\langle \sigma^2 \rangle \equiv C_0 \delta l \equiv \langle \sigma(\mathbf{r})^2 \rangle_0 \equiv \frac{1}{\Omega} \sum_{\mathbf{q}} \frac{1}{\gamma |\mathbf{q}|^2}
\]  

(13)

Here \( \langle \sigma^2 \rangle \) is \( O(\delta l) \) since the sum runs over a shell of thickness \( \sim \delta l \).
E. Cumulant terms (see G, sec. 12.3)

1. First-order cumulant term (see G, sec. 12.3)

   Here and in the next section, we focus on the terms of \( O(\delta l) \) since only they will contribute to the R.G. flow in the limit \( \delta l \to 0 \).

   Plugging (7c) into the first-order cumulant, we get

   \[
   \langle \hat{V} \rangle_0 = \int \frac{1}{2} \langle \sigma^2 \rangle (3uS_\rho^2) + \frac{1}{4} \langle (\sigma(r)^4) \rangle_0 
   \]

   including all the polynomial terms. Of course, only terms even in \( \sigma \) survive, since \( \sigma(r) \) fluctuates symmetrically around 0. (For a general function \( U(S) \), the factor in parentheses in (14) would be \( U''(S_\rho(r)) \).

   there is no eq. (15) or (16)

   In the second cumulant term \( \frac{1}{2} \langle \hat{V}^2 \rangle_0 \), the term with the smallest power of \( \sigma \) is

   \[
   \int_{r_1} \int_{r_2} \frac{1}{2} U'(S(r_1))U'(S(r_2)) \langle \sigma(r_1)\sigma(r_2) \rangle_0.
   \]

   The next smallest power appears in

   \[
   \int_{r_1} \int_{r_2} \frac{1}{2} \left[ \frac{1}{2} U''(S(r_1)) \right] \left[ \frac{1}{2} U''(S(r_2)) \right] \langle \sigma(r_1)^2\sigma(r_2)^2 \rangle_0
   \]

   \[
   = \int_{r_1} \int_{r_2} \frac{1}{2} \left[ \frac{1}{2} U''(S(r_1)) \right] \left[ \frac{1}{2} U''(S(r_2)) \right] (2G_{>}(r_1 - r_2)^2 + \langle \sigma^2 \rangle^2)
   \]

   using Wick’s theorem to obtain the last line. (A term in \( \langle \sigma(r_1)\sigma(r_2)^3 \rangle_0 \) is possible too, but that becomes \( 3\langle \sigma^2 \rangle G_{>}(r_1 - r_2) \) using Wick, and thus its fate is similar to (17)).

   The general form in (17) and (17’) and some others is

   \[
   \int_{r_1} \int_{r_2} \frac{1}{2} f(S_\rho(r_1))f(S_\rho(r_2))G_{>}(r_1 - r_2)^m
   \]

   where \( f(S) \) is some function (some derivative of \( U(S) \)).

   Here is where we make the big leap. Expressions like (18) are nonlocal, unlike anything in the original \( U(S) \); how can we mash them back into the prescribed shape? Well, we notice that \( G_{>}(R)^m \) is decaying with distance \( |R| \) and so (hopefully) can be approximated as some multiple of a delta function, \( C_m \delta l \delta(R) \) where

   \[
   C_m \delta l \equiv \int_R G_{>}(r)^m
   \]
P HYSICS 653  Lec. 2.7: Momentum-space R.G.  P.9

[This all can be justified better by making Taylor expansions around the midpoint: see supplement.] In defining this with the δl factor, we were anticipating – see Eqs. (24) – that this constant is proportional to the thickness of the momentum shell.

The full integrand in (18) now factors into an integral over \( \mathbf{F} \) and one over \( \mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2 \), yielding

\[
C_m \delta l \int d^2 \mathbf{F} (S_<(\mathbf{r}))^2
\]

(20)

There are no eqs (21)-(23)

Integrals of \( G_>(R) \)

Now, \( Gg(r) \) is a very wiggly function and most of the \( C_m \) will cancel. You can show that the spatial averages of \( G_>(R) \) are

\[
\int G_>(R) = 0 \quad \text{(24a)}
\]

\[
\int R^2 G_>(R) = 0 \quad \text{(24b)}
\]

\[
\int G_>(R)^2 \equiv C_2 \delta l = \frac{1}{12} \sum_\rightarrow \frac{1}{(\gamma q^2)^2} = O(\delta l)
\]

(24c)

\[
\int G_>(R)^3 = O(\delta l^2) \quad \text{(24d)}
\]

[EXERCISE (c). Verify (24a,c,d).]

[EXERCISE Estimate the amount of phase space in the double sum over wavevectors that (24d) gives, to verify that the volume of overlap scales as \( O(\delta l^2) \).]

Thus \( C_m = 0 \) except for \( C_2 \), and the 2nd order contribution we care about is from (17'):

\[
\mathcal{H}_{\text{sum}} = ... + \int d^d \mathbf{r} \frac{1}{8} C_2 \delta l \ U''(S_<(\mathbf{r}))^2
\]

(25)

Note (24c) is the same integrand which gives a divergence if we tried to integrate it over all \( \mathbf{q} \) “in one fell swoop.” But here the integral is harmless, since we are just nibbling away a thin spherical shell – the domain of integration goes nowhere near \( \mathbf{q} = 0 \) where the integrand blows up.

Finally we can collect the function of \( S_\subset \) from first and second order and put it in the original form.

F. Rescaling

Eq. (9) or (10) is not quite the desired new Hamiltonian \( \mathcal{H}_{\text{new}} \), since the new degrees of freedom \( S_\subset \) are not yet quite like the old ones \( S \), namely the Fourier content of the variables in \( \mathcal{H}_{\text{sum}} \) only extends up to the cutoff \( \mathbf{q} = \Lambda/b \) not to \( \mathbf{q} = \Lambda \). We must rescale the space, \( \mathbf{r}_{\text{new}} = \mathbf{r}/b \), \( \mathbf{q}_{\text{new}} = b \mathbf{q} \), so as to make \( \Lambda_{\text{new}} = \Lambda \).
Thus we put

\[ S_{\text{new}}(q) \equiv b^{2x_S} S_\varsigma(q/b) \]

i.e.

\[ S_{\text{new}}(r) = b^{2x_S} S_\varsigma(br) \]  \hspace{1cm} (26)

where at the moment \( b^{2x_S} \) is just a constant to be determined. The arbitrary overall scale factor \( b^{2x_S} \) in (26) is fixed by a convention: we force \( \gamma' = \gamma \), so that the only change is in the shape or size of \( U(S) \).

1. Re-scaling and exponent \( \eta \).

We can relate \( x_S \) to the exponent \( \eta \) using (29) and (30). Recall

\[ \langle S(r)S(r') \rangle_0 \sim |r - r'|^{-(d-2+\eta)} \]  \hspace{1cm} (27)

at criticality (\( \sim \) Lec. 2.2). At the fixed point (27) must be true before and after we transform; furthermore, it involves large \( r - r' \) properties so it relates to the \( S_\varsigma \)'s. Thus we conclude that (27) ought to be invariant under (26). But the left hand side of (27) is bilinear in \( S_\varsigma \)'s, so it must rescale by \( b^{2x_S} \), while the right hand side must rescale as \( b^{d-2+\eta} \). Consequently we find (generally)

\[ 2x_S \equiv d - 2 + \eta \]  \hspace{1cm} (28)

In other words, the correlation function decays as \( 1/r^{2x_S} \).

By counting how many powers of \( \nabla \) and \( S \) each term has, the change of variables is seen to be: (EX)

\begin{align*}
at_{\text{new}} &= b^{d-2x_S} at_{\text{sum}} \hspace{1cm} (29a) \\
\gamma_{\text{new}} &= b^{d-2-2x_S} \gamma_{\text{sum}} \hspace{1cm} (29b) \\
u_{\text{new}} &= (b^{d-4x_S} u_{\text{sum}} \hspace{1cm} (29c)
\end{align*}

In the Gaussian model, \( u_{\text{sum}} = u \), \( \gamma_{\text{sum}} = \gamma \), and \( at_{\text{sum}} = at \). But we fix \( \gamma_{\text{new}} = \gamma \). That forces \( x_S = \frac{1}{2}(d-2) \) for the Gaussian model, and \( x_t = 2x_S \) and \( x_u = 4x_S \). In other words, the scaling exponents are simply proportional to the power of \( S \) which appears.

That’s what people mean when they talk of “power counting”: finding the (trivial) scaling exponents of a mean-field approach. The reason this is pertinent is those are the correct exponents in \( d = 4 \), and we’re working near to \( d = 4 \) so they are close to the actual exponents. In particular, if a term is irrelevant in \( d = 4 \), it’s likely to be irrelevant close to \( d = 4 \) (the R.G. eigenvalue can’t flip sign immediately.)

G. Result: R.G. recursion

We now start to put together all the above results. Following through the coefficients, we obtain

\[ at_{\text{sum}} = at + 3\langle \sigma^2 \rangle - O(u^2) \]  \hspace{1cm} (30a)
where the $O(u)$ term comes from $O(\hat{V})$ and the $O(u^2)$ terms came from $O(\hat{V}^2)$; we won’t bother with the coefficients of the latter, since they won’t affect the $O(\epsilon)$ R.G. Thus $\langle \sigma^2 \rangle$ and $\frac{\Delta}{G^2}$ were the only factors we need to know.

Similarly,

$$\gamma_{\text{sum}} = \gamma - O(u^2) \quad (30b)$$

where there was no contribution from $O(\hat{V})$, and the $O(u^2)$ terms from the $O(\hat{V}^2)$ cumulant do not affect the $O(\epsilon)$ R.G. Finally,

$$u_{\text{sum}} = u - O(u^2) \quad (30c)$$

Specifically:

$$u_{\text{sum}} = u - 9C_2u^2 \quad (30c')$$

with $C_2$ from (24c)

**[EXERCISE (d). Find the coefficient of $u^2$ in (30a)...]**

Finally, the only contribution to the quartic term (25c) comes from (18e); this is the term we desired (recall (17b')) in order to get the nontrivial Wilson-Fisher fixed point.

**[EXERCISE (e). By spatially averaging the kernels in (23), as in Ex. (b), verify (25c).]**

**[EXERCISE (f). Use the fact that the $O(u^2)$ term in (25b) has form $-C_\gamma (\Delta \delta) u^2$ to find $\eta$ in terms of $u_*$ and $C_\gamma = \gamma$ (to $O(u^2)$). This is really the most trivial of the “exercises”.]**

Now we apply the general notions from Sec. 2.7 F. We’re following the convention of fixing $\gamma_{\text{new}} = \gamma$. And (25b) said that, to our order in the expansion, we get $\gamma_{\text{sum}} \equiv \gamma$. Then (29b) implies

$$x_S = (d - 2)/2 \quad (30)$$

By comparing that to (27') we see $\eta = 0$. (We would need to work out $\gamma_{\text{new}}$ to $O(u^2)$ terms if we wanted to compute $\eta$ to $O(\epsilon^2)$ as quoted in eq. (0b).)

**RG recursion relations (this is a sort of Appendix)**

Now, after the rescaling (29) is inserted into eqs. (30),

$$a_{\text{new}} = b^2(at + 3\langle \sigma^2 \rangle u) = b^2(at + \frac{3C(d)}{(\gamma q^2)^2}u) \quad (31a)$$

$$u_{\text{new}} = b^{4-d}(u - 9G^2 u^2) = b^\epsilon u(1 - \frac{9C(d)u}{(\gamma q^2)^2}) \quad (31b)$$

where $\epsilon \equiv 4 - d$. In (31),

$$C(d) \equiv \Lambda^d K_d (1 - b^{-d}) \quad (32)$$

where $K_d$ is the factor you get when converting spherically symmetric integrals to radial coordinates, $\int d^d k/(2\pi)^d \rightarrow K_d \int k^{d-1}dk$. 
[EXERCISE (g). Verify (32).]

Indeed (31a) flows away from (either) fixed point: $t$ is a relevant variable as it should be. The second term in (31b) allows a nontrivial fixed point – the “Wilson-Fisher fixed point” – at a small nonzero value of $u_*$. In fact it will turn out that $u_*$ is $O(\epsilon)$ when $\epsilon \equiv 4 - d$. This is why there is an expansion in powers of $\epsilon$. We are really doing an expansion in powers of $u_*$, which is small only near $d = 4$. In turn $t_*$ will have a nonzero value, and the exponents will differ from the trivial ones by amounts of order $u_*$ i.e. of order $\epsilon$.

Finally, letting $\delta l \to 0$, we can convert (31) into a pair of differential recursion relations (see HW.) From that, we will obtain $\nu$ to $O(\epsilon)$, as advertised; since we already found $\eta$ to $O(\epsilon^2)$, we have all the exponents.