Ex. 2.4.4 Localization Length for a Low-Resistance Wire

(a)
To find the mean free path, we first express the conductance as:
\[ \sigma = \frac{L}{(RA)} = \frac{L}{(0.2L \times 9 \times 10^{-8} \Omega m^2)} = 7.07 \times 10^{-2} (\Omega \cdot \text{nm})^{-1}. \]
Now, noting that \( n = \frac{8\pi}{3} (k_F/2\pi)^3 = 85 \text{nm}^{-3} \) yields \( k_F = 13.6 \text{nm}^{-1} \), we have:
\[ \sigma = R_Q^{-1} \left( \frac{2\pi}{13.6 \text{nm}^{-1}} \right) (85 \text{nm}^{-3}) l_F; \quad l_F = 46.5 \text{nm}. \]
Finally, since \( g(L) = R_Q \sigma_0 L^d \), using the values provided above we have \( g(L) = 1.82 \times 10^{12} L^{-1} \text{ m}^{-1} \).

(b)
Our flow equation is
\[ \frac{d \log g}{d \log L} = -1 \quad (g > 1). \]
If we let \( f = \log g \) and \( x = \log L \), we quickly see that \( f = -x + C \) where \( C \) is a constant. If we then choose as our initial condition \( C = \log R_Q \sigma_0 \) we obtain \( g(L) = R_Q \sigma_0 / L \), and therefore \( g(L^*) = 1 \) when \( L^* = R_Q \sigma_0 \). Note that this matches the standard bulk result in 3d; the resistance of a wire is proportional to its length, and therefore the conductivity should be inversely proportional to the length of the wire.

(c)
Now assume the scaling relation
\[ \frac{d \log g}{d \log L} = -1 + \log g \quad (g < 1). \]
Again let \( f = \log g \) and \( x = \log L \). The solution to this differential equation is
\[ f = ce^x + 1; \quad g = e^{e^{L+1}} \]
Since this equation is only valid for \( g < 1 \), \( L \) is a positive real number and \( g(L^*) = 1 \), we must have \( c = -1/L^* \).
We now imagine that the same copper is a sheet of thickness $\theta = 0.1\text{mm}$. Given $\sigma_{3d}$ from part (a), we have

$$\sigma_{2d} = \theta \sigma_{3d} = 1.825 \times 10^8 R_Q^{-1}. \quad (6)$$

The scaling relation in 2d is

$$\frac{d \log g}{d \log L} = -\frac{1}{g}. \quad (7)$$

Using the same method we used in (a-c) and requiring that $g(\theta) = R_Q \sigma_{2d}$, we have

$$g(L) = -\log L + c; \quad c = \log \theta + R_Q \sigma_{2d}. \quad (8)$$

Solving for $g(L^*) = 1$ gives $L^* = 8.9 \times 10^7\text{m}$; an enormously larger value than in 3d and a stark demonstration of why length scales can usually be ignored in measurements of 2d conductivity.

Ex. 3.1.3 Electronic States of the Fibonacci Chain

(a) First consider the process of generating Fibonacci chains: the substitutions $S \rightarrow L$ and $L \rightarrow LS$. Note that both types of sites in the original chain ‘k’ give an $L$ in the new chain ‘k + 1’, and only the $L$ sites in $k$ yield $S$ sites in $k + 1$. Hence the number of $S$ and $L$ sites satisfy the recurrences

$$n_{L_{k+1}} = n_L^k + n_S^k$$
$$n_{S_{k+1}} = n_S^k + n_L^{k+1} = n_S^k + n_L^k = N_k + N_{k-1}. \quad (9)$$

Further, if chain 0 is $S$, and hence chain 1 is $L$, then we have $N_0 = N_1 = 1$. Thus $N_k = F_k$, the Fibonacci numbers. The solution to this recurrence relation can be found exactly as in the previous homework, by trying exponential solutions etc. to get

$$F_k = \left(\tau^k + \left(-\frac{1}{\tau}\right)^k\right)/2,$$

with $\tau = (1 + \sqrt{5})/2$, the golden ratio (root of the characteristic polynomial $x^2 = x + 1$).

In the reduction shown in Figure 3.1.6, we are reducing wells with 2 $L$’s to a site $L'$ and those with a single $L$ to a site $S'$. This can be achieved by the substitutions $LLS \rightarrow L'$ and $LS \rightarrow S'$. Hence the reduced chain (b) has as many sites as the number of $S$ sites in chain (a). Thus the reduction in number of sites is:

$$\frac{n(b)}{n(a)} = \frac{N_k}{n_k^2} = \frac{N_k}{N_k - n_k^2} = \frac{N_k}{N_k - N_{k-1}} = \frac{F_k}{F_{k-2}} \rightarrow \tau^2 \text{as } k \rightarrow \infty$$

Hence $a' = \tau^2 a$, where $\tau$ is the golden ratio.

(b) The Hamiltonian consists of hopping $t$ between neighbouring $L$ sites, $T$ between $L$ and $S$ sites, and a potential $W$ on the $S$ sites. In order for a 1 parameter RG analysis to work, we will need $T$ and $t$ to be in a definite proportion, as we will discover below. In the regime of $t, T \ll W$, this causes the $L$ sites to look like wells bounded by the $S$ sites. To lowest order the particles don’t hop over the barriers. This amounts to solving the problem inside each well, assuming infinite barrier potential.
The wells with a single $L$ site have $H_{\text{well}}^L = 0$ and a ground state energy of 0 in this approximation, while those with two $L$ sites have a hopping Hamiltonian between those sites $H_{\text{well}}^{LL} = \begin{pmatrix} 0 & -t \\ -t & 0 \end{pmatrix}$ which has eigenvalues $\pm t$, and a ground state energy $-t$. Thus the $LL$ wells are lower in energy by $t$ with respect to the $L$ wells. Hence the $LL$ wells are converted to $L'$ sites and the $L$ wells to $S'$ sites in the reduced chain. The difference in the effective potential is $W' = t$.

There are two types of hopping to be considered, from an $LL$ well to an $L$ well, and between two $LL$ wells, with an $S$ site being the intermediate state in each case. For $t \ll W$, the initial and final states for the hopping are approximately degenerate and we get $\langle L_1|H|L_2 \rangle = -\langle L_1|H|S \rangle \langle S|H|L_2 \rangle / (E_S - E_{L_1} - E_{L_2})$. It is important to note that for an $LL$ well, there would be a factor of $1/\sqrt{2}$ from the amplitude for the particle to be at the $L$ site from/to which hopping occurs. This leads to $t' = t_{LL} = t^2/W\sqrt{2}$, while $t' = t_{LL+LL} = T^2/2W$. Note that $T = t'\sqrt{2}$, and the new chain ‘looks’ like the old one only if $T = t\sqrt{2}$. If this were not true, we would have to do a 2-parameter renormalization flow. With this constraint, we get $t' = t^2/W$. Note that $t \ll W \Rightarrow t' \ll W'$, and hence this reduction process can be repeated.

Now we have two similar models, one with parameters $(t, W)$ and the other with $(t', W')$. For a mode of wavenumber $k$ (with respect to sites of that model, not the physical wavevector), the energy $E(k, t, W) = tE(k, W/t)$, since the Hamiltonian is linear in $t$ and $W$, and any non-trivial dependence is only with respect to the ratio $W/t$. In this case, $W'/t' = t/(t^2/W) = W/t$. Hence $E'(k')/E(k) = X = t'/t = t/W$.

(c)

The wave function should have the same physical wavevector as shown in Figure 3.1.6. Since the lattice spacing is increased by a factor of $\tau^2$ in the reduced chain, the corresponding wavenumber increases by the same factor. Hence we must have $E'(k') \approx E(k)$ with $k' = \tau^2 k$.

(d)

Combining the results of (b) and (c), we get $E(k) = E'(\tau^2 k) = X E(\tau^2 k)$ or $E(\tau^{-2} k) = X E(k)$, for any $k$. For a sequence $k_n = \tau^{-2n} k_0$, we get $E(k_n) = X^n E(k_0) = \text{const.} (\tau^n)^{\log X} \propto k_n^2$ with $z = -\frac{\log X}{2\log \tau}$. If we require that $E(k)$ be smooth, this would lead to $E(k) \propto k^2$.

(e)

We used perturbation theory to get $t'$ and $W'$ from $t$ and $W$. This required the assumption $X = t/W \ll 1$. Hence taking $W \to 0 \ (X \to \infty)$ is not valid.

Ex. 4.1.5 Quantum spin pairs

(a)

The Hamiltonian can be rewritten as follows:

\[ H = H_0 + H' = |J| \langle \vec{s}_1 \cdot \vec{s}_2 - g_\mu_B H \cdot (\vec{s}_1 + \vec{s}_2) \]

\[ = |J| \left( \langle \vec{s}_1 + \vec{s}_2 \rangle^2 - \langle \vec{s}_2 \rangle^2 / 2 - g_\mu_B H \cdot (\vec{s}_1 + \vec{s}_2) \right) \]

\[ = |J| \langle \vec{S}^2 / 2 - g_\mu_B HS_z \rangle - |J| s(s+1)(\text{if } \vec{H} \parallel \hat{z}, \text{and } s_z^2 = s_2^2 = s^2) \]

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Since \([S^2, S_z] = 0\), we have \([S^2, \mathcal{H}] = 0\) and \([S_z, \mathcal{H}] = 0\). Hence any state with definite \(S\) and \(S_z\) are eigenstates.

The possible values of \(S\) are 0, 1, 2, \ldots, \(2s\) when \(s^2 = s^2_z = s^2\). The energy for an \((S, S_z)\) eigenstate is clearly \(E = |J|S(S+1)/2 - g\mu_B HS - |J|s(s + 1)\).

\(S_z = +S\) gives the lowest energy for each \(S\), and hence \(E_{\text{min}}(S, H) = |J|S(S+1)/2 - g\mu_B HS - |J|s(s + 1)\), with a corresponding magnetization \(\vec{M} = (g\mu_B S)\hat{z}\).

Pretending that \(E_{\text{min}}(S, H)\) is a continuous function of \(S\), energy is minimum for \(S = (g\mu_B H/|J| - 1/2)\) - this suggests that the optimum \(S\) increases with \(H\). To account for discrete \(S\), look at the value of \(H\) for which \(E_{\text{min}}(S, H) = E_{\text{min}}(S+1, H)\) which occurs for \(H = |J|(S+1)/g\mu_B\), and at this \(H\), \(S + 1\) becomes the spin which minimizes \(E\). Hence the magnetization is \(m(H) = (g\mu_B H/|J|)\), while classically it would be \(m(H) = g\mu_B H/|J|\). The classical and quantum graphs do have the same average slope, and they both saturate at \(m = 2s\).
Ex. 4.1.6 Magnetization Step Data

In Figure 4.1.3, each peak corresponds to a step change in the magnetization as in exercise 4.1.6, except that the steps are not sharp but rounded. The $\frac{dm}{dB}(B)$ data (extracted by vectorizing figure 4.1.3) is fit to a series of $\text{sech}^2 \left( \frac{B-B_{\text{centre}}}{B_{\text{width}}} \right)$ peaks (which should model thermal broadening adequately):

The fit reveals peaks centered around $(10.65 \pm 0.02)T$, $(19.79 \pm 0.02)T$, $(29.66 \pm 0.03)T$, $(38.51 \pm 0.04)T$ and $(48.94 \pm 0.04)T$. A linear regression analysis yields that the spacing in $B$ between successive peaks is $(9.53 \pm 0.13)T$. Using the results of Exercise 4.1.6, we then deduce that $|J|/g\mu_B = (9.53 \pm 0.13)T$, and hence $|J| = (1.102 \pm 0.015) \times 10^{-3} \text{eV}$. 

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Ex. 4.1.7 Triangles of Spins

To deal with the Hamiltonian with three spins interacting in a triangle, we again rewrite the Hamiltonian in terms of total spin. Let \( \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 \). Noting that \( \mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3 = 1/2 ( \mathbf{S} \cdot \mathbf{S} - \mathbf{S}_1 \cdot \mathbf{S}_1 - \mathbf{S}_2 \cdot \mathbf{S}_2 - \mathbf{S}_3 \cdot \mathbf{S}_3 ) \), our Hamiltonian is

\[
H = \frac{J}{2} ( \mathbf{S} \cdot \mathbf{S} - \mathbf{S}_1 \cdot \mathbf{S}_1 - \mathbf{S}_2 \cdot \mathbf{S}_2 - \mathbf{S}_3 \cdot \mathbf{S}_3 ) - g\mu_B B \hat{z} \cdot \mathbf{S}. \tag{1}
\]

Since the magnetic coupling is negative, the ground state is once again when \( S_z = S \) and \( s_1 = s_2 = s_3 = s \). In this state the energy spacing will again be in terms of \( S, S_z \) and \( s \), exactly as in the previous problem. The energies are

\[
E = \frac{J}{2} (S(S + 1) - 3s(s + 1)) - g\mu_B B S_z. \tag{2}
\]

The magnetization steps will again occur at linearly spaced increments depending on the value of \( S \), and since this spacing is the same as in the previous case (the \( S^2 \) and \( S_z \) terms are identical), there is no way to differentiate a material with pairs of spins from one with triangles of spins from examining the magnetization alone.