Critical Ising Spin Dynamics on Percolation Clusters

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Ferromagnetically interacting Ising spins are placed on a fractal network (such as a percolation cluster) with \( T_c = 0 \), and endowed with a single-spin-flip dynamics. At low temperatures the collective dynamics is determined by thermal activation over energy barriers. The barrier to overturning of the spins in a domain of size \( L \) is proportional to \( Z \ln L \), where \( Z \) is a new geometrical parameter characterizing the fractal. A new "singular" dynamic scaling is found in which the effective dynamic critical exponent diverges at criticality.

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In recent years, there has been much study of physical problems in a space which, instead of being Euclidean, is a fractal lattice, such as a percolation cluster.\textsuperscript{1} This has included the study of the equilibrium properties of the ordering of spin systems\textsuperscript{2}—a problem which is inherently nonlinear—and also of linear dynamic problems, such as random walks,\textsuperscript{3,4} dielectric response,\textsuperscript{3} elastic modes,\textsuperscript{3,4} and spin waves.\textsuperscript{4}

However, there had been no study of the critical dynamics near the ordering temperature of a fractal lattice, a nonlinear and dynamic phenomenon. Recently, Aeppli, Guggenheim, and Uemura\textsuperscript{6} addressed this problem by an inelastic neutron scattering experiment on \( \text{Rb}_2(\text{Mg}_{0.41}\text{Co}_{0.59})\text{F}_4 \), where the magnetic Co ions are at percolation on a two-dimensional lattice. They fitted a relaxation time \( \tau \) by the standard dynamic scaling form,\textsuperscript{7} \( \tau (T) \sim \xi_T (T)^2 \), with an exceptionally large value of \( z \) (here \( \xi_T \) is the thermal correlation length). Very recently, theoretical approaches\textsuperscript{8,9} have appeared which assume this form and then proceed to obtain estimates of \( z \).

In this Letter, I consider the behavior of ferromagnetic Ising systems on finitely ramified fractal lattices, such as the incipient infinite cluster in percolation ("percolation cluster" for short, or PC). Other examples are Sierpiński gaskets\textsuperscript{11} or PC's on a Bethe lattice.\textsuperscript{12} This analysis turns out to be unexpectedly simple since this system has a zero-temperature phase transition and this means that the most divergent factors in \( \tau (T) \) can be derived by consideration of thermal activation over energy barriers resulting from the nonuniform geometry. As a function of the length scale \( L \), the barrier \( E_{\text{max}} \) grows logarithmically:

\[
E_{\text{max}}(L)/2J \approx Z \ln L + \text{const} \quad \text{as} \quad L \to \infty \quad (1)
\]

Here \( Z \) is a new, "universal" parameter of the fractal. I also show that (1) implies a "singular" dynamic scaling, in which \( \tau (T) \) diverges as \( \exp(\text{const}/T^2) \), faster than any power of \( \xi_T (T) \).

I start with a Hamiltonian describing classical Ising spins \( s_i \) placed on the \( N \) sites of the PC in any dimension \( d \geq 2 \) with ferromagnetic nearest-neighbor couplings \( J \):

\[
H = -J \sum_{i<j} s_i s_j. \quad (2)
\]

The static critical behavior is known: Percolation clusters are "finitely ramified,"\textsuperscript{12} so that \( T_c = 0 \). With a temperature variable \( \theta = e^{-2J/T} \), the thermal correlation length\textsuperscript{13,14} is given by

\[
\xi_T (T) \sim e^{2\nu_p \sqrt{T}/T} a \sim \theta^{-\nu_p} a, \quad (3)
\]

where \( \nu_p \) is the percolation correlation-length exponent and \( a \) is the lattice constant.

I assume a single-spin-flip (hence, spin-nonconserving) dynamics satisfying the usual detailed balance condition, e.g., Glauber dynamics\textsuperscript{15}: spins flip at a rate \( \sim e^{-\Delta E/\tau \vartheta_0^{-1}} \) (energy increased by \( \Delta E \)) or \( \sim \tau_0^{-1} \) (energy decreased or unchanged by flip). Then \( \tau (L, T) \) is defined as the typical (median-among-realizations) lifetime of the slowest relaxing mode of a piece \( \mathcal{P} \) of PC with diameter \( L \).\textsuperscript{16} The characteristic time of the infinite system is \( \tau (T) = \lim_{L \to \infty} \tau (L, T) \sim \tau (\xi_T, T) \).

The first task is to derive Eq. (1). Take \( L \ll \xi (T) \). It follows that \( \mathcal{P} \) spends most of its time in its two ferromagnetic ground states with occasional transitions over the barrier separating them. Thus

\[
\tau (\mathcal{P}, T) \sim e^{E_{\text{max}}(\mathcal{P})/T \vartheta_0}. \quad (4)
\]

where \( E_{\text{max}}(\mathcal{P}) \) is an energy barrier—the highest energy on the lowest path connecting the ground states of \( \mathcal{P} \) in its configuration space. The entropy term in the exponent of Eq. (4) can be neglected, compared to \( E_{\text{max}}/T \), as long as \( L \ll \xi_T \).

To evaluate \( E_{\text{max}}(\mathcal{P}) \), I will use the links-and-blobs model of \( \mathcal{P} \)'s\textsuperscript{17} (see Fig. 1). The PC can be pictured as a (contorted) chain of one-dimensional "links," interrupted by "decorations": multiply connected "blobs" in the chain, or "dangling ends" extending from it. The dangling ends and the blobs are made up of similarly decorated chains in a self-similar fashion. Outturning \( \mathcal{P} \) corresponds to moving a domain wall along the chain. On sections of purely one-
with threefold coordination, and I conjecture that it is true for arbitrary clusters. The other optimal sequences are elaborations of such a sequence by temporary backtracking, side excursions, or alternative routes around states in the noncritical part \( E < E_{\text{max}} \) of the sequence.

Let us make the scaling assumption that \( E_{\text{max}}(\mathcal{P}) \) depends only on the diameter \( L(\mathcal{P}) \). Then \( \mathcal{B}' \) is the subblob with the largest diameter \( L' \). The Hermann-Stanley\(^\text{19}\) subblob distribution law implies \( L' \sim b^{-1}L \), where \( b > 1 \) is the typical ratio of diameters of a blob to its largest subblob. Substituting into (5) yields

\[
\epsilon_{\text{max}}(L) = 1 + \epsilon_{\text{max}}(b^{-1}L),
\]

the solution of which is (1) with

\[
Z = (\ln b)^{-1}.
\]

I have neglected the possibility that \( \mathcal{B}' \) was a dangling end, so that the above argument actually gives \( Z \) for the backbone of the PC.\(^\text{19}\) In fact a rule similar to (4) governs dangling-end barriers,\(^\text{12} \) and so (1) and (7) should still hold for the full PC, with \( b(\text{full PC}) < b(\text{backbone}) \).

The argument leading to (1) and (7) becomes exact on deterministic self-similar networks, which are generated by stages in which each link is replaced by a piece of decorated chain. In particular, \( b = 3 \) for the nonrandom "squig" (see Mandelbrot and Gaven,\(^\text{20}\) Fig. 2) which models the full PC in \( d = 2 \) (\( b = 3 \) for the squig backbone, too) and \( b = 4^{\text{1}}/\text{3} \) on a hierarchical lattice modeling the PC backbone in \( d = 2 \).\(^\text{21}\)

Furthermore, Eq. (1) is clearly a consequence of hierarchical geometry and hence ought to hold on any finitely ramified fractal lattice. It has been proved that (1) holds for the Sierpiński gasket,\(^\text{11}\) with \( Z = 1/2 \ln 2 \), and for PC’s on Bethe lattices,\(^\text{12}\) with \( Z = 2/\ln 2 \). (Note that these lattices are, respectively, all blobs and all dangling ends.) In fact PC’s in dimension \( d \geq 6 \) have the same geometry as PC’s on Bethe lattices,\(^\text{12}\) and so (1) is essentially proved for PC’s for \( d \geq 6 \).

The parameter \( Z \) for PC’s is universal in that it depends only on \( d \). I emphasize that \( Z \) is not in general a function of the Hausdorff (fractal)\(^\text{1} \) or spectral (fraction) dimensions,\(^\text{3,4} \) or of the percolation exponents; it seems to be yet another of the many geometrical parameters needed to characterize a fractal.\(^\text{1} \)

I have ignored the contribution of very rare but very slowly relaxing compact clusters—regions which happen to be undiluted because of statistical fluctuations—which give rise to a "Griffiths phase" below the \( T_c \) of the pure system.\(^\text{22}\) Consider the probability distribution of barriers \( \epsilon_{\text{max}}(\mathcal{P}) \) in the ensemble of \( \mathcal{P} \’s \) with diameter \( L \). As a result of the compact clusters it has a tail \( \sim L^d \exp(-c_{\text{max}}^{\text{undiluted}}L^{1-d}) \) extending far beyond \( \epsilon_{\text{max}}(L) \). The largest such barrier in a typical domain of diameter \( L \) goes as \( \epsilon_{\text{max}}^{\text{undiluted}} \sim C(d) \).
Thus \( \tau(L, T) \sim L^{\xi(T)} \), where \( \xi(T) \) diverges as \( T \to 0 \), in contrast to standard dynamic scaling.

Equation (8) is valid only for \( L \ll \xi_T \). To get \( \tau \sim L^{\xi(T)} \), \( \xi(T) \), it is necessary to include finite-temperature corrections, i.e., entropy effects. The most important of these turns out to be the existence of many alternative paths with the same, optimal barrier (e.g., configuration B' instead of B in Fig. 1), so that relaxation is faster than the naïve substitution \( L \to \xi_T \) in (8).

Scaling can give the form of this correction. The static thermodynamic functions scale with \( \xi_T \) and PC's are self-similar (at \( L \geq a \)), so that

\[
\tau(a, T) = F(L/\xi_T(T)) \tau(T),
\]

with \( \tau(a, T) \approx \tau_0 \). Now Eq. (8) can be rewritten with use of (3):

\[
\ln \tau(L, T) \approx Z v_p^{-1} \ln \xi_T \ln L - C \ln \xi_T
\]

for small \( \ln L \). (I am now setting \( \tau_0 = a = 1 \).) This can be reconciled with the form (9) only if there is an additional term of \( O(\ln L)^2 \) in (10), with

\[
\ln F(x) = -\frac{1}{2} v_p^{-1} Z (\ln x)^2 + O(\ln x)
\]

and

\[
\ln \tau(T) \approx \frac{1}{2} v_p^{-1} Z (\ln \xi_T)^2 + O(\ln \xi_T)
\]

\[
= 2 v_p Z (J/T)^2 + O(J/T) + \ldots ;
\]

so finally

\[
\ln \tau(L, T) \approx Z \left[ -\frac{1}{2} v_p^{-1} (\ln L)^2 + (2J/T) \ln L \right] + O(J/T, \ln L).
\]

The result (13) is contrasted with ordinary dynamic scaling\(^7\) in Fig. 2. It is valid for \( L \leq \xi_T \); when \( L \sim \xi_T \) additional entropy corrections to (13) appear.

These results for \( \tau(L, T) \) can be better understood by considering the nature of the dynamic renormalization group (RG)\(^7,22\) which would give rise to them. Assume that we have a one-parameter, discrete static RG. Rescaling length by the natural rescale factor, \( b \), we have \( \xi_T' = \xi_T/b, \theta' = f(\theta) \approx b^{l/v} \theta \), with \( f(0) = 0 \) as the relevant fixed point. The simplest way to extend this to a dynamic RG has one time parameter \( \tau \). To give (9), its rescaling must have the form

\[
\tau' = \lambda(\theta) \tau.
\]

This is consistent with (12) only if

\[
\lambda(\theta) \sim \theta
\]

for small \( \theta \). Ordinary dynamic scaling is also characterized by an RG equation like (14) except that in place of (15), ordinarily \( \lambda(\theta) \rightarrow b^{-\xi} \), a constant, as \( \theta \to 0 \). Hence I propose the term “singular dynamical scaling” for the behavior (15). Note that if an approximate RG is constructed\(^10\) which violates (15), it cannot give the correct behavior in the \( T \to 0 \) limit.

A somewhat different dynamic RG has been constructed for the Sierpinski gasket.\(^11\) This renormalization group uses a master-equation approach: The dynamics is formulated as a random walk (with weights for each step) in configuration space, a diffusion problem which is equivalent to a resistor network. The network is self-similar and can be solved exactly, provided that only optimal paths are considered. This allows exact calculation of some non-trivial entropy corrections, which are similar in outline to those found here, but different in detail since the static RG is singular in this case\(^2\) with \( \theta' = \theta + O(\theta^2) \).

Other random systems with \( T_\text{r} = 0 \) have a dynamics described by energy-barrier activation (4), but of course singular dynamic scaling follows only if the barriers scale logarithmically. McMillan’s phenomenological dynamic RG for an Ising spin-glass at its lower critical dimension (i.e.,\(^24\)) exhibits singular dynamic scaling, with a recursion like (15) leading to \( \ln \xi_T \sim T^{-2} \) and \( \ln \tau \sim T^{-3} \). On the other hand, in the random-field Ising model\(^25\) at \( d = 2 \), the lower critical
dimension, the barriers scale as a power of $L$.

What does (12) imply for experiments? One signature of singular dynamic scaling is upward curvature in a $(\ln \xi_T, \ln \tau)$ plot. In a small temperature range about a given $T$, the data will be consistent with the effective exponent $\tilde{z} = d \ln \tau / d \ln \xi_T \sim T^{-1}$, this suggests why the neutron experiment$^6$ should find an anomalously large value for $z$. Actually the temperature range of Ref. 6 should be large enough to see curvature, but it does not. This is easily rationalized since $\xi_T$ was still only $\sim 3a$ at the lowest $T$ studied.

If simulations, or a repeat of the experiment at lower $T$, can reach the scaling regime ($\xi_T \sim L^*)$, the curvature in (12) should be visible. Also, inelastic neutron scattering as a function of momentum transfer $q$ could test (13). The best numerical way to check Eq. (1) would be exact evaluation of $\epsilon_{\text{max}}$ for randomly generated PC’s.


9J. H. Luscombe and R. C. Desai, to be published.
11C. L. Henley, unpublished.
12C. L. Henley, unpublished.
16Take the diluted lattice at percolation and cut out any (hyper)cubed of side $L$. Then let $\phi$ be the largest connected cluster (which, typically, spans the cube).
21L. de Arcangelis, S. Redner, and A. Coniglio, Phys. Rev. B 31, 4725 (1985). Note that $L$ is not naturally defined for their lattice, and so I have taken $L = L_1^{\frac{1}{\nu}}$ where $L_1$ is the number of “links” in the hierarchical lattice and $\nu_\phi = \frac{4}{\nu}$ is the percolation exponent for genuine $d = 2$ PC’s.