Huse, Henley, and Fisher Respond: Here we show how the exponents $\zeta = \frac{4}{7}$ for the transverse fluctuations in interface position and $\chi = \frac{1}{7}$ for the fluctuations in the free energy can be derived exactly for an interface in a random potential in two dimensions at any temperature. We do this by relating the problem of the interface to the damped Burgers’s equation$^1$ in one dimension with random forcing, the scaling behavior of which has been analyzed by Forster, Nelson, and Stephen$^2$.

In the continuum limit with Hamiltonian$^3$

$$ H = \int dx \left[ \frac{1}{2} \sigma (\partial y/\partial x)^2 + V(x,y) \right], $$

(1)

the weight $W(x,y)$ of a path or interface ending at $(x,y)$ satisfies the equation

$$ \frac{\partial W(x,y)}{\partial x} = \frac{k_B T}{2\sigma} \frac{\partial^2 W(x,y)}{\partial y^2} + \frac{1}{k_B T} V(x,y) W(x,y), $$

(2)

where $\sigma$ is the interface stiffness, $y(x)$ is the location of the interface, and the correlations in the random potential are

$$ \langle V(x,y)V(x',y') \rangle = \Delta \delta(x-x') \delta(y-y'). $$

This is the continuum version of Kardar’s$^4$ recursion relation for the weights in the lattice solid-on-solid (SOS) model. If we define $u(x,y) = \partial F(x,y)/\partial y$, where the free energy is $F(x,y) = -k_B T \ln W(x,y)$, Eq. (2) becomes

$$ \frac{\partial u(x,y)}{\partial x} = \frac{k_B T}{2\sigma} \frac{\partial^2 u(x,y)}{\partial y^2} - u(x,y) \frac{\partial u(x,y)}{\partial y} $$

$$ - \frac{1}{\sigma} \frac{\partial V(x,y)}{\partial y}, $$

(4)

which is Burgers’s equation$^1$ with a diffusion constant or damping proportional to $T$ and conservative random forcing, $\partial V/\partial y$. When (4) is viewed as a nonlinear diffusion equation, $x$ serves as the time coordinate and $y$ as the space coordinate, and $u(x,y)$ is the drift velocity. That $u(x,y)$ is indeed a velocity, which scales as distance over time $(y/x)$, is necessary because of the Galilean invariance of (4). The free energy $F(x,y)$ has a term that is linear in $x$. Since $u = \partial F/\partial y$, however, the fluctuations in $F$ about this average value scale as $y^2/x$. The fluctuations in $F$ scale as $x^2$ and $y$ scales as $x^4$, and so this implies $\chi = 2\zeta - 1$. This exponent relation was pointed out by Huse and Henley$^3$ and can also be seen by examining the gradient-squared term in the Hamiltonian (1).

The forced Burgers’s equation (4) obeys a fluctuation-dissipation theorem$^2$ as a consequence of which its steady-state distribution is simply

$$ P[u(x,y)] \propto \exp\left[ -\frac{1}{2} \lambda \int dy \ u^2(x,y) \right], $$

(5)

with $\lambda = \sigma k_B T/\Delta$. This invariant distribution implies that

$$ \langle [F(x,y) - F(x,y')]^2 \rangle = \sigma |y - y'|/\lambda, $$

(6)

and, hence, $2\chi = \zeta$. The two exponent relations together dictate $\zeta = \frac{4}{7}$ and $\chi = \frac{1}{7}$, which are equivalent to the exponents derived by Forster, Nelson, and Stephen$^2$ for (4). The analysis of Forster, Nelson, and Stephen$^2$ implies that, for a given $\lambda$, the same fixed point governs the behavior of (4) at large distance and time scales for all $\Delta$, including in the limit $T \to 0, \Delta \to 0$ at fixed $\lambda$. This limiting case of the Burgers’s equation with neither forcing nor damping is exactly integrable.$^1$ The scaling exponents discussed above were first obtained by Burgers,$^4$ who studied the evolution in this integrable limit of random initial conditions with a distribution similar to (5).

Kardar and Nelson$^3$ recently solved a model of parallel interfaces with disorder and hard-core repulsion, from which they indirectly obtained the exact exponents $\zeta$ and $\chi$. A similar scaling behavior has also been found by van Beijeren, Kutner, and Spohn,$^7$ for a hard-core lattice-gas model of one-dimensional conduction.

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$^1$J. M. Burgers, The Nonlinear Diffusion Equation (Reidel, Boston, 1974).


$^4$M. Kardar, preceding Comment [Phys. Rev. Lett. 55, 2924(C) (1985)].

$^5$U. Deker and F. Haake, Phys. Rev. A 11, 2043 (1975). Note that if one makes the natural extension of (1), describing a string in a random potential, to higher dimension, the fluctuation-dissipation theorem no longer holds for the corresponding generalization of the forced Burgers’s equation (4) and there is not a known invariant distribution. See also Ref. 2.
