Lecture 6.6

Vortex lattice state

In a “Type II” superconductor, meaning the ratio of penetration depth to coherence length $\lambda/\xi > 1/\sqrt{2}$, magnetic field can partially penetrate a sample, but not in the form of coexisting normal-state domains characteristic of Type I superconductors (Sec. 6.3 C). Instead, in Type II, since a domain’s wall energy is negative (see Sec. 6.3 B), it “wants” to be as small as possible, which (thanks to flux quantization, Lec. 6.4) amounts to a vortex carrying a quantized flux $\Phi_0^* \equiv \hbar c/e^* = \hbar c/2e$ and quantized circulating current. Consequently, for a range of applied fields $H_{c1} < H < H_{c2}$, the equilibrium state consists of a regular lattice of the (mutually repelling) vortices, called the Abrikosov phase (or sometimes, misleadingly, the “mixed state”). Fig. 3(f) in Lec. 6.0 already showed the (mean-field, disorder-free) phase diagram.

This lecture works through the equilibrium properties of this phase. (The equally important transport properties are left to Lec. 6.7.)

The stages of the story are

- The symmetry breaking and elastic theory of the vortex lattice (analogous to an ordinary crystal, but of lines rather than pointlike atoms) in Sec. 6.6 A. It spontaneously breaks both translational symmetry (vortex positions) and gauge symmetry (phase of $\Psi(\mathbf{r})$).

- The radial dependence of $\Psi(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$, and $\mathbf{J}_s(\mathbf{r})$ around a vortex (Sec. 6.6 B, plus its line tension and effective potential of interaction with nearby vortices (Sec. 6.6 C) – all this analogous to the domain walls in Lec. 6.2.

- The basic phase diagram (Sec. 6.6 E, and the ($T = 0$) values for the lower critical field (Sec. 6.6 W), at which the first vortices enter, and the upper critical field (Sec. 6.6 X), at which the superconducting order parameter goes completely away.

Lec. 6.7 will continue the story. Vortex motion controls the electrical conductivity and the magnetic relaxation. In fact, a superconductor in the Abrikosov state is truly superconducting only if the flux lines are pinned. To understand pinning, we revisit the phase diagram, now taking thermal fluctuations and pinning into account (Sec. 6.7 Y), as are especially pertinent to high-$T_c$ cuprates. One phase, the entangled vortex liquid, is illuminated by a clever mapping to a Bose condensed liquid in two dimensions (Sec. 6.7 Z).
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(a). triangular lattice  \quad (b)

\[ H_{\text{app}} \]

Figure 6.6.1: Abrikosov vortex lattice. (a). side view; the lines indicate vortices, and the arrows indicate the sense of flux which they carry. \textit{[Sorry, the little disks here are only to highlight the \((x,y)\) positions of the vortices.} (b). viewed down the magnetic field. Arrow shows circulation of supercurrent, and star marks the singularity in the vortex core where \(|\Psi| \to 0\). Dashed line shows Wigner-Seitz cells, and an integration contour is marked (heavy) around one of them; the current is zero along these lines.

**Topological property and vortex “charge”**

A vortex is an example of a “topological defect” (see Sec. 1.5X and the comparative table there). That means its existence can be inferred from measurements along a loop encircling the vortex without probing the region around the vortex core; and there is no way to eliminate it, except by annihilation with a vortex of the opposite sense.

The “vortex charge” \( n \) is defined by a dependence \( e^{in\phi} \) replacing the function in (6.6.4), which was the special case \( n = +1 \). \((\Delta\theta = 2\pi n \text{ when we traverse it counterclockwise})\). An antivortex (relative to these axes) is equally possible, with a phase factor \( e^{i(-\phi+n)} \), that is \( n = -1 \). The name “charge” is used because

(i) it is conserved – the only way to get a vortex out of your system is to push it to a boundary, or annihilate it with an antivortex;

(ii) it acts as a source for other fields.

\textit{Indeed, the vortex charge \( n \), phase field \( \theta(\mathbf{r}) \), current field \( \mathbf{J}_s(\mathbf{r}) \), and vortex-vortex interaction are respectively analogous to the electric charge, potential, field, and Coulomb interaction in electrostatics.} \[\text{SHOULD I SAY THIS LATER IN AN EXPLICIT TABLE?}\]

Vortices with \(|n| > 1\) are not found, since the energy is reduced if they split up into single vortices. (See Ex. 6.6.2.) Opposites attract and tend eventually to annihilate; but when the vortices are induced by an external applied field, they all have the same sign, and like repels like.

### 6.6 A Vortex lattice: structure, symmetry breaking

Imagine it: an ordering not of material particles, but of topological defects in the abstract field \( \Psi(\mathbf{r}) \) representing the subtle kinds of electron correlations that characterize superconductivity. Yet the vortices are as real as the tiles on your bathroom floor, and they were imaged decades before the advent of scanned-probe microscopy. The lattice symmetry is triangular, like most other 2D problems (e.g. the Wigner crystal), since that’s the best packed lattice; in the present case, for a given density of vortices, the triangular lattice has the largest inter-vortex distance.
6.6 A. VORTEX LATTICE: STRUCTURE, SYMMETRY BREAKING

In $^4$He, a similar vortex array develops in a sample in a rotating bucket; putting it in a non-inertial frame is analogous to putting the metal in a magnetic field.

**Position order**

As usual for a new kind of ordered state, we will discuss its order parameter, symmetry, and elasticity. There are two symmetry breakings in the ideal Abrikosov lattice: vortex position order and phase order.

Position order means the locations of the vortex lines, like atoms in an ordinary solid, and is the more important kind of order in a vortex lattice: these are the asterisks in Fig. 6.6.1 (b). This breaks translational symmetry in the $x$ and $y$ directions like a (two-dimensional) crystal. As in Lec. 1.4, we can define an order parameter

$$\Phi_k(\rho(z), z) = e^{ig\rho(z)}$$ (6.6.1)

Its lattice constant is $a \sim 10$ nm in classic superconductors, enormous compared to atomic spacings (and $a$ diverges at $H_{c1}$).

The position order has been observed experimentally in (at least) three ways in superconductors (see Fig. 6.6.2), not counting the vortex arrays in rotatintg neutral superfluids.

1. The vortex array diffracts neutrons (their magnetic moments couple not only to spin but orbital moments) giving Bragg peaks in the structure factor (at small angles, since the intervortex spacing is large)

2. The superconductor’s surface is decorated with a nanoscale magnetic powder, which (not unlike more macroscopic iron filings) adheres preferentially where the $B$-field is strongest thus marking vortex cores. This powder adheres after the sample is warmed up, forming a sort of photograph which is read by scanning electron microscopy.

3. Finally, vortices are observed by tunneling microscopy. (The tunneling rate is sensitive to the local value of the gap in the quasiparticle dispersion relation, which vanishes at the vortex center which is in effect normal.)

**Phase order**

A second kind of symmetry breaking and long-range order is gauge symmetry – the phase $\theta(r)$ of the order parameter $\Psi(r)$, as in the plain Meissner phase. We take (for definiteness) the London gauge ($\nabla \cdot A = 0$; (Somewhat surprisingly, in that gauge $\theta(r)$ is practically uniform throughout space.) To see this is an independent symmetry breaking, note one could multiply $\Psi(r)$ by any uniform phase factor without changing the pattern of $|\Psi(r)|$. Thus, the Abrikosov phase maintains superconducting order even while the field partially penetrates.

Each kind of order has its kind of elasticity. The coarse-grained $\theta$ elasticity is related to the coarse-grained $\theta$ as in the bare Ginzburg-Landau $F_{\text{grad}}$, except that a averaged $n_s$ should be used. In Lec. 6.7, the phase elasticity will be related to the background supercurrent, whereas the $|\Psi|$-lattice elasticity is pertinent to the (de)pinning behavior.

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3In principle, thermal fluctuations of the vortex positions destroy long-range phase order at any $T > 0$; however, the correlation length is over 1 mm. [M. A. Moore, Phys. Rev. B 45, 7336 (1992)].
Imagine deforming the vortex lattice on scales much larger than its lattice constant. These are shears and compressions of the vortex array and have their own continuum elastic theory, analogous to that of an ordinary solid. However, here the displacement field $u(r)$ has just two components $u = (u_x, u_y)$: a lengthwise displacement of a vortex line does not make a distinct state; its argument still has three components, $r = (x, y, z)$. The explicit form of free energy is

$$F_{\text{elast}} = \frac{1}{2} \int d^3r \left\{ K_z \left| \frac{\partial u(r)}{\partial z} \right|^2 + 2\mu \sum_{ab} u_{ab}(r)^2 + \lambda (\nabla_{xy} \cdot u(r))^2 \right\}$$

(6.6.2)

where

$$u_{ab} \equiv \frac{1}{2} \left( \frac{\partial u_a}{\partial p_b} + \frac{\partial u_b}{\partial p_a} \right)$$

(6.6.3)

for $a, b = x, y$, and $\nabla_{xy} \cdot u \equiv u_{xx} + u_{yy}$. Here $K_z \propto \epsilon_c$ is the tilt stiffness, $\lambda$ is the in-layer bulk (compression) modulus, and $\mu$ is the (much smaller) in-layer shear modulus.

What about the displacement Goldstone mode, associated with the continuous symmetry breaking of the vortex positions? In real life, the motion of a vortex in a superconducting metal is highly overdamped, not only due to random pinning, but also to Ohmic-like dissipation by the electrons. Hence the vortex lattice has no sound mode, but only an overdamped mode is expected.

“Pancake vortices”

[Possibly this belongs in a different place]

One extreme case of type II superconductor (approximated by many cuprates) consists of a stack of two-dimensional superconducting layers with no interlayer Josephson

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4See Lec. 6.7 for more about this. Note also that although an $s$-wave superconductor has a gap to electronic excitations in its bulk, the vortex core is like a one-dimensional metal and supports gapless excitations. This should be discussed in Lec. 7.4 on the Bogoliubov-de Gennes equations, even in $^4$He, an utterly pure material, vortex lines are damped by emitting phonons, I believe, as they move relative to the fluid.
6.6 B. ONE VORTEX

A “pancake vortex” is confined to one layer (and so called because its extent in that layer is much larger than the layer spacing). Despite the lack of phase coherence, the magnetic energy suffices to align the vortex positions in adjacent layers (at $T = 0$): otherwise, flux must travel horizontally between the layers connecting one vortex to the next. However, when there are thermal or pinning fluctuations and a high vortex density, it may no longer be possible to say which vortex in one layer corresponds to which one in the next layer: see Sec. 6.7 Y for more.

**Vortex lattice as soft matter**

Several other inhomogeneous phases are periodic arrays of discrete defects in a medium:

1. Incommensurate modulations of a crystal, such as charge-density waves (Lec. 3.4); more closely particular, the Frenkel-Kontorova model (Lec. 3.3) forms a one-dimensional lattice of “discommensuration” defects; the onset of this lattice (the commensurate-incommensurate transition) is quite reminiscent to the lower transition to the Abrikosov lattice at $H_{c1}$ (Sec. 6.6.23).

2. The Wigner crystal (Lec. 1.4 W [omitted]) of (charged) electrons in a semiconductor, or on the surface of liquid helium.

3. Arrays of charged stripes in cuprates (Lec. 8.3) like the high-$T_c$ superconductors.

4. The array of steps on a crystal surface with a slight overall tilt relative to a facet direction.

These arrays have an elastic theory arising out of interactions which are (typically) mediated by the underlying medium’s own elasticity, e.g. the wall-wall interactions in the Frenkel-Kontorova model. They almost always have overdamped (not propagating) sound modes, as the defects (or particles) experience viscous drag from the surrounding medium. Since the above lattices have periods much larger than the interatomic spacing, they should (under a driving force) drift smoothly relative to the underlying lattice, but in practice they tend to get pinned by the medium’s disorder (inhomogeneities and non-topological defects). Pinning of the vortex lattice is discussed in Lec. 6.7, in Sec. 6.7 W.

[Alternative introduction of elastic theory.] If a continuous symmetry is broken, then one has an elastic theory. This is the analog of the elastic theory of an ordinary solid, if we take a flux line as being analogous to an atom. This exists at a coarser-grained scale than the elastic theory of the $\Psi(\mathbf{r})$ field itself. In that sense it is a quite independent theory from the GL elastic theory; it is dependent, in the sense that the flux lattice elastic constants are functions of the GL parameters and of the lattice constant $a$.

**6.6 B  One vortex**

A vortex in a neutral or charged superfluid, is a line-like topological defect in the order parameter phase. We will first consider the neutral case before going on to the charged (superconducting) case. In the latter case, a vortex additionally has one flux quantum $\Phi_0$ bound to it.

\[ Another related example is a colloidal solid, formed in liquids by particles with $\mu$m diameter (and comparable lattice constants). \]
Vortices are also called "flux lines" in superconductors since they carry a quantized magnetic flux.\(^6\)

**Fields and currents around a vortex**

Consider a normal domain with magnetic field through it. We can integrate the vector potential along a loop within the bulk of the superconducting portion that surrounds the normal domain, just as we did for a ring with an empty center (in Lec. 6.4). So the total flux through the domain is an integer multiple of the flux quantum \(\Phi_0\). The vortex is a domain with just one \(\Phi_0\): vortices are quantized.

Since magnetic field lines can’t terminate, the vortex must be a line-like object. The minimum energy configuration of fields and currents has cylindrical symmetry around the singular line, as shown in Fig. 6.6.3(a).

Figure 6.6.3: (a). Field \(B(r)\) and current \(J_s(r)\) around a flux line. Recall that \(J_s(r)\) is essentially the (gauge-invariant) gradient of the order parameter phase angle. (b). Radial dependence of order parameter amplitude \(|\Psi|\) and magnetic field \(B\) as a function of radial distance from a flux line.

The spatial dependence of the order parameter is

\[
\Psi(r, \phi, z) = f(r)e^{i(\phi + \chi)} \tag{6.6.4}
\]

where \((r, \phi, z)\) is \(r\) in cylindrical coordinates (with vortex is centered at \(r = 0\)), and \(\chi\) is an arbitrary phase. The boundary conditions are \(f(0) = 0\) and \(f(\infty) = \Psi_0\).\(^7\) The detailed shape of \(f(r)\) depends on \(\lambda/\xi\). The radial dependences, shown in Fig. 6.6.3(b), look qualitatively like those of the normal-superconductor domain wall as a function of \(x\) (see Fig. 6.3.3). A vortex looks just like a wall that has been wrapped around an axis!

The tube-like region in which \(|\Psi(r)| \approx 0\) is the vortex "core" and has radius \(O(\xi)\), according to the Ginzburg-Landau theory.\(^8\) The core radius (in which \(|\Psi(r)|\) is suppressed) is only \(\sim \xi\) (see Fig. 6.6.3(a) again), whereas supercurrent \(J_s\) and \(B\) decay with \(r\) over the larger length \(\sim \lambda\).

The magnetic field attached to a vortex, in the charged superfluid, is thus a secondary consequence of the currents (by Ampère’s law). The associated vector potential \(A(r)\) grows with radius from the vortex line. Since the current is actually proportional to the gauge-invariant derivative \(\nabla_\Lambda \theta\), it follows that \(A(r) = \nabla \theta(r)\) at large \(r\): that is,

\(^6\)The alternate term "flux line" is somewhat misleading since the vortex is defined by the order parameter field. A vortex has a precisely defined line along which \(|\Psi(r)|\) vanishes and \(\theta(r)\) is undefined. Similar quantized lines and vortex lattices exist in neutral superfluids, but nothing like flux is bound to them.

\(^7\)If \(f(0)\) were nonzero, then \(|\nabla \Psi| \sim 2\pi f(0)/r\) near \(r = 0\) due to the angle dependence in (6.6.4), and the \(F_{\text{grad}}\) energy term would diverge.

\(^8\)Which gets the core wrong when \(T \ll T_c\). See, perhaps, Lec. 7.8 [omitted] about the relation of GL to BCS theory.
the currents around a vortex get _screened_ for \( r > \lambda \) [As mentioned in Lec. 6.2, this is analogous to the screening of a charge in a metal at distances beyond the [Debye] screening length.]

## 6.6 C Vortex field configuration and line energy

The _LONDON LIMIT_ is about the same as _EXTREME TYPE II LIMIT_ \( \lambda / \xi \gg 1 \), which is the simplest situation, since \( n_s \) is a constant and there’s just one length scale \( \lambda \) at play.

### Neutral case

First consider a _neutral_ superfluid. The superflow is just \( \nabla \theta \) (rather than \( \nabla A \theta \)), and so a vortex has \( J_s(r) \propto 1/r \) at large \( r \).

\[
|\nabla \theta| = \begin{cases} 
0 & \text{for } r < \xi \\
1/r & \text{for } \xi < r < L 
\end{cases} \quad (6.6.5)
\]

where \( R \) is the container size. The vortex energy per unit length is proportional to the integral of \( 1/r^2 \) in this case, which diverges as \( \ln(R/\xi) \).

Note a small part of the cost for a vortex of length \( L \) comes from the “core”, a cylindrical region of radius \( \sim \xi \), in which the system is basically normal, \( |\Psi(r)| \approx 0 \). The cost for doing this is the condensation energy \( |F_{\text{cond}}| \) times the “normal volume” \( \sim \pi \xi^2 L \) : thus

\[
\epsilon^\text{core}_c \approx c_{\text{core}} |F_{\text{cond}}| \xi^2 \quad (6.6.6)
\]

where \( c_{\text{core}} \) is a constant. \(^9\)

The mathematical structure of the fields around a vortex is quite analogous to the N-S interface problem (in Lec. 6.3). \(^10\)

Let us recall,

\[
F_{\text{grad}} = \frac{\hbar^2}{2m_s} |(\nabla - i e_s \hbar c A(r)) \Psi(r)|^2; \quad (6.6.7)
\]

Let’s assume the London limit \( n_s = |\Psi|^2 = \Psi^2_0 \) (Sec. 6.2 B) outside the core, i.e. almost everywhere since \( \xi \) is small. Then the phase gradient energy becomes

\[
F_{\text{grad}} = \frac{\hbar^2}{2m_s} n_s |\nabla A - \theta(r)|^2, \quad (6.6.8)
\]

where

\[
\nabla A \theta \equiv \nabla \theta - \frac{e_s}{\hbar c} A(r). \quad (6.6.9)
\]

Also, recall

\[
J_s(r) = \frac{\hbar}{2m_s} n_s \nabla A \theta \quad (6.6.10)
\]

Let \( \Phi_{\text{in}}(r) \equiv \int dr 2\pi r dr B(r) \) be the flux within radius \( r \), and \( \Phi_{\text{out}}(r) \equiv \Phi_0^* - \Phi_{\text{in}}(r) \) be the rest of the flux. Once again, we recapitulate the line integral tricks from

\(^9\)Due to the variation of \( |\Psi| \) with \( r \), there is a contribution from the \( \nabla |\Psi| \) term in \( F_{\text{grad}} \) which also is \( O(|F_{\text{cond}}| \xi^2) \).

\(^{10}\)In the wall (free) energy (Lec. 6.3), the result was a multiple of the condensation energy \( |F_{\text{cond}}| \) times \( O(\lambda) - O(\xi) \). Here, in the line energy, \( \lambda \) and \( \xi \) appear not to the first but to the zero-th (as in logarithmic) order: namely, our result comes out proportional to \( |F_{\text{cond}}| (\ln \lambda - \ln \xi) \).
Sec. 6.4 C, with variations. Obviously, \( \oint \nabla \theta \cdot dr = 2\pi \), and by the usual Stokes identity \( \oint A \cdot dr = \Phi_{\text{in}}(r) \); inserting into (6.6.10) gives
\[
|\nabla_A \theta(r)| = \frac{\Phi_{\text{out}}(r)}{r} \frac{1}{\Phi_0}
\] (6.6.11)
where \( \Phi_{\text{out}}(r) = \Phi_0 - \Phi_{\text{in}}(r) \) is the flux outside radius \( r \). Thus, \( J_s(r) \propto |\nabla_A \theta| \) gets diminished by flux inside that radius, but it doesn’t vanish, so long as \( \Phi_{\text{in}}(r) < \Phi_0 \). Meanwhile, Ampère’s law says \( dB(r)/dr = (4\pi/c)J_s(r) \); thus \( B(r) \) doesn’t vanish either, so long as \( J_s(r) \) doesn’t. With increasing \( r \), there continue to be screening currents until the full flux quantum is achieved, \( \Phi_{\text{in}}(r) \approx \Phi_0 \). That is the screening: at larger \( r \), the vector potential exactly cancels the gradient in (6.6.10) so the gradients are zero.)

**Crude estimate**

We can now estimate the energy of a vortex crudely. Most of the \( B \) field is inside radius \( \lambda \); hence, at \( r > \lambda \), most of the gradient in (6.6.10) is canceled by the vector potential – the first ratio in (6.6.11) is small and, roughly speaking,

\[
\nabla_A \theta \approx \begin{cases} 
0 & \text{for } r < \xi \\
1/r & \text{for } r < \lambda \\
0 & \text{for } r > \lambda 
\end{cases}
\] (6.6.12)

if we considered only the gradient term, dividing the total cost by the vortex line’s length \( L \), we get

\[
\epsilon_v = \frac{F^{\text{tot}}_{\text{grad}}}{L} = F_{\text{cond}} \xi^2 \int_{\xi}^{\lambda} (2\pi r dr) \frac{1}{r^2} \sim |F_{\text{cond}}| \xi^2 \ln(\lambda/\xi)
\] (6.6.13)

for \( \lambda \gg \xi \); the log divergence was cut off and replaced by \( \ln(\lambda/\xi) \).

In a charged superfluid, of course, we must add the field energy contribution: it comes out equal to the gradient term, merely multiplying our estimate (6.6.13) by two.

*I need to insert! See calculation in Tinkham Sec. 5.1; it’s a bit less straightforward to see where the log comes from in this term.*

**Deriving exact \( B(r) \) profile around a vortex**

*incomplete derivation*

Back when the penetration depth \( \lambda \) was introduced in Lec. 6.2, our starting point was

\[
\nabla^2 B(r) + \frac{1}{\lambda^2} B(r) = 0
\] (6.6.14)

which gave a plain exponential decay \( B \sim e^{-z/\lambda} \) in a one-dimensional geometry. Eq. (6.6.14) also governs the decay of magnetic field with \( r \), after we replace

\[
\nabla^2 \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}
\] (6.6.15)

for cylindrical coordinates. You can check that the asymptotic behavior must be \( f(r) = r^{-1/2}e^{-r/\lambda} \); that function satisfied

\[
\nabla^2 f(r)/f(r) = (1/\lambda^2)
\] (6.6.16)
plus corrections of order $1/r^2$.

The exact result for $B(r)$ (in the London approximation) is proportional to $K_0(r/\lambda)$, where $K_0()$ is a zero order modified Bessel function, with

$$K_0(r/\lambda) \approx \begin{cases} 
\ln(\lambda/r) & r \lesssim \lambda; \\
\frac{\pi}{2r} e^{-r/\lambda} & r \gg \lambda.
\end{cases}$$  

(6.6.17)

at large distances.\(^{11}\) (Which gives the matching of short and long coefficients, that I could not have obtained from my asymptotics.)

The total formula with the correct coefficient is

$$\epsilon_v \approx \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \ln\left(\frac{\lambda}{\xi}\right) = 4\pi F_{\text{cond}} \xi^2 \ln(\lambda/\xi) = 4\pi \frac{\hbar^2}{2m_e} n_s \ln(\lambda/\xi).$$  

(6.6.18)

The second formula on the right hand side of (6.6.18) required the coherence length definition, $(\hbar^2/2m_e\xi^2)n_s\equiv F_{\text{cond}}$.

**Estimate of vortex line cost $\epsilon_v$ and its temperature dependence**

To continue the analogy to a domain wall, which had a surface energy $\sigma_W$, a vortex has a line energy (per unit length) $\epsilon_v$. Its importance is (i) it enters the tradeoff in free energy costs determining $H_{c1}$; (ii) it gives the line tension which determines an elastic constant of the Abrikosov phase; (iii) the energy barrier to pass one vortex line across another can be guessed\(^ {12}\) as $\sim \epsilon_v\xi$: this is an activation energy in vortex dynamics at nonzero temperature. (See Lec. 6.7; it is probably the nucleation barrier for making vortex line in the phase slip argument of Lec. 6.4.)

We can numerically estimate the vortex line energy in terms of fundamental constants. Notice the final form in (6.6.18) didn’t involve $\xi$ or $\Psi$ or (to anticipate Part 7) the gap $\Delta$: it only depends on $n_s/m_e$ which is (roughly) the ratio of the free electron density and effective mass.

Now recollect the energy scales of a typical metal as treated in the simplest free-electron approximation. The Fermi wavelength is $2\pi/k_F \sim 0.1\text{nm}$; the Fermi energy is $E_F \sim \hbar^2 k_F^2/2m_e$, assuming $m_e \sim m_e$; and the electron density is $n \sim k_F^3$. Inserting all this into (6.6.18) we get

$$\epsilon_v \sim \frac{\hbar^2 k_F^2}{2m_e} k_F \sim E_F k_F \sim 10\text{eV/nm}$$  

(6.6.19)

(ignoring the logarithm factor in (6.6.18), which will typically be of order unity). That’s a large energy. Need to check the number in (6.6.19).

We’d expect the vortex line energy to vanish at $T_c$, since that is the point where there is no difference between the superconductor and the normal metal. Well, $\Psi_0 \sim (T_c-T)^{1/2}$ hence $\epsilon_v(T) \sim n_s(T) \sim (T_c-T)$. Another way to think of this last formula is that $\epsilon_v \propto 1/\lambda(T)^2$. So $\epsilon_v$ indeed vanishes as $T \rightarrow T_c$. We’d have to get extremely close to $T_c$ in order for the vortex-line-crossing barrier ($\sim \epsilon_v\xi$) to be less than the temperature $T$. But the reduction of $\epsilon_v$ does increase the transverse thermal fluctuations, with important consequences for the phase diagram beyond mean-field theory (See Sec. 6.7 Y).

\(^{11}\)de Gennes, page 63.

\(^{12}\)Seeing that the affected length of one line would be comparable to the core diameter of the other line. [This could be a T.Q.].
Note: Even in high-$T_c$'s with $\lambda/\xi > 100$, the log factor in (6.6.18) is only $\approx 5$. Modulo the log, (6.6.18) has a crude interpretation that $v_v$ is the condensation energy cost of driving the core to the normal state.  

6.6 D Vortex-vortex interaction

**Bending energy: vortex wandering term**

The framework from Sec. 6.6 C can also be deployed to compute an effective Hamiltonian for a vortex configuration. Unlike an atom in a crystal, a single vortex line is an extended object with an internal degree of freedom:

The path of each vortex is given, to an excellent approximation, by a single-valued function $\rho(z)$, consisting of the two coordinates transverse to $z$. Whereas the state space of an ordinary 3D crystal of $N$ atoms is parametrized by $3N$ arbitrary coordinates, here the state space of $N$ vortices is parametrized by $2N$ arbitrary functions of $z$, so this system is somewhat more complicated to describe.

Our first step must be to determine the one-vortex wandering term. In an isotropic superconductor, it is simply the arc length times the energy per unit length (string tension) $v_v$ of a vortex, a term which resists tilting the vortex lines away from vertical,

$$U_{tilt} = \int dz \epsilon_v \sqrt{1 + \left( \frac{d\rho}{dz} \right)^2} \approx \epsilon_v L + \int dz \frac{1}{2} \epsilon_v \left| \frac{d\rho}{dz} \right|^2$$  \hspace{1cm} (6.6.20)

We define $\mathbf{l}$ oriented along the average path of the vortex. (The sense is chosen by the right hand rule, such that the supercurrent $\mathbf{J}_s(\mathbf{r})$ runs counterclockwise, when viewed down $\mathbf{l}$.)

**Effective vortex-vortex Hamiltonian**

The next step is the pair interaction of two vortex lines. Two straight vortex lines oriented along the field ($z$) axis, and separated by a distance $r \gg \xi$, feel an effective repulsive potential

$$V_{eff}(r) = \frac{\Phi_0^2}{8\pi^2 \lambda^2} K_0(r/\lambda)$$  \hspace{1cm} (6.6.21)

(free energy per unit of length in the $z$ direction). The effective interaction (6.6.21) could be derived by constraining a system to have vortices at (say) $x = \pm r/2$, and then finding the configurations $\Phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ which minimize the total Ginzburg-Landau free energy, in the same spirit as the calculations that found the free energy of the N-S domain wall, or the core energy $v_v$ of one vortex.  

13 The vortex-vortex interaction, mediated by the elasticity of the superconducting order parameter between vortices, is completely analogous like the elastically mediated interaction of crystal defects in Sec. 1.5Y.

14What about a vortex running in the sense opposite to the external field? It has an additional term $\Phi \mathbf{i} \cdot H$ per unit length in the $z$ direction. This strongly disfavors even a bit of back-tracking in a vortex.

15Indeed, $K_0()$ in (6.6.21) is the same function appearing in the profile of a single vortex. The interaction is essentially the product of the fields due to one vortex with the sources of the other vortex's fields, as is typical of all mediated interactions: we first same that for interactions mediated by the Fermi sea, Lec. 1.3 X .
The short distance logarithmic behavior is the same form as the repulsion between lines of electric charge. In the case of a neutral superfluid, the logarithm is valid out to infinity; the exponential factor in (6.6.21). (It was due to screening of the field by supercurrent, and the cancellation of $A(r)$ and $\nabla \theta(r)$ terms in the gauge-invariant gradient, so the circulation decays with distance; these are specific to charged superfluids.)

In the dilute regime near $H_{c1}$, at least, we can approximate the vortex-vortex interaction by (6.6.21). Then we can write an effective Hamiltonian of the entire vortex array as

$$
\mathcal{H}_{\text{eff}} = \sum_i \int dz \epsilon_v \sqrt{1 + \left( \frac{\partial \mathbf{\rho}}{\partial z} \right)^2} + \sum_{i<j} \int dz \ V_{\text{eff}}(\mathbf{\rho}_i(z) - \mathbf{\rho}_j(z))
$$

$$
\equiv \int dz \mathcal{H}_{2D}(\{\mathbf{\rho}_i(z)\}) \quad (6.6.22)
$$

The first term in (6.6.22) is just (6.6.20) and the second term is (6.6.21). Eq. (6.6.22) is a good model description, but it is approximate in that it includes only interactions between vortices at the same $z$ coordinate.

To relate this to the elastic free energy of (6.6.2), first write $u = \mathbf{\rho}_i(z) - \mathbf{\rho}_i^{(0)}(z)$. The first term of (6.6.22) gives rise to $z$ shear elasticity (terms proportional to $|\partial u/\partial z|^2$, while the second term gives rise to shear and compression elastic terms quadratic in $\partial u/\partial x$ and $\partial u/\partial y$.

### 6.6 E Mean-field phase diagram

**TO MELD.** In applied fields $H < H_{c1}$, the superconductor has a “Meissner” phase in which flux is perfectly excluded, just like a Type I superconductor below $H_c$. Above $H_{c2}$, it enters the normal metal phase in which flux penetrates completely, just like a Type I superconductor with $H > H_c$. We can understand $H_{c1}$ and $H_{c2}$, respectively, as instabilities of the superconducting and normal states; details are left to Secs. 6.6 W and 6.6 X.

The Abrikosov phase (see Fig. 6.6.1) consists of a regular array of parallel vortex lines, which (viewed down the axis) form a triangular lattice. Each line is associated with exactly one flux quantum, so the mean magnetic field $\overline{B}$ is related to the lattice constant by $\Phi_0/Ca^2$, where $Ca^2$ is the unit cell area and the coefficient is $C = \sqrt{3}/2$ for a triangular lattice.

The “thermodynamic” critical field $H_c$ is defined by $F_{\text{cond}} = H_c^2/8\pi$, as in Type I superconductors, but nothing happens at this value of applied field. It turns out that

$$
H_{c1} \approx \frac{\ln \kappa}{\sqrt{2\kappa}} H_c \quad (6.6.23)
$$

and

$$
H_{c2} = \sqrt{2\kappa} H_c \quad (6.6.24)
$$

where $\kappa \equiv \lambda/\xi$ is the Ginzburg-Landau parameter so $H_c$ is roughly the geometric mean of the two critical fields. $[H_{c1}$ is the field at which flux first enters the superconductor in equilibrium, and $H_{c2}$ the point at which it goes completely normal.]

---

\[16\] Anisotropic Type II superconductors are more complicated in that (i) the flux lattice is distorted away from triangular symmetry (ii) if $H$ is not aligned to a symmetry direction, then $\overline{B}$ will not be aligned with $H$ either (even though the sample is needle-shaped and aligned with $H$.)
A vertical dashed line in (a) marks the places where $|\Psi| \to 0$, but was meant to be cut out. Note that case (a) only occurs just above $H_{c1}$. The Meissner phase has $(|\Psi|, B) = (0, 0)$ and the Normal phase has $(|\Psi|, B) = (0, H)$.

The relation to $\xi$ and $\lambda$ is derived simply by using the identities relating Ginzburg-Landau parameters (from Lec. 6.1). It’s easy to remember the result:

$$H_{c1} \sim \ln \kappa \Phi_0^2/\lambda^2$$

and

$$H_{c2} \sim \Phi_0^2/\xi^2.$$  

(6.6.25)

(6.6.26)

The $|T_c - T|$ critical behavior shown in Fig. 3(f) simply reflects that of $H_c(T)$; since $\xi$ and $\lambda$ both behave as $|T - T_c|^{-1/2}$, $\kappa$ remains essentially temperature-independent.

We can also notice that (if we forget the logarithm in (6.6.25)!) $H_{c1} = H_{c2} = H_c$ according to the above formulas, exactly when $\kappa = 1/\sqrt{2}$; for smaller $\kappa$ values, $H_{c1}$ would be larger than $H_{c2}$, so what happens instead is the first-order transition directly from the Meissner to the normal state (at $H_c$). This criterion for the critical value of $\kappa$ agrees exactly with that one (Lec. 6.3) that the interface free energy $W$ vanishes — that seems rather deep to me, since there are not any walls at all in the calculations of either $H_{c1}$ or $H_{c2}$ (see sections below).

Both transitions are continuous: the penetrating flux $|\mathbf{B}|$ (averaged across a unit cell) increases from zero to $H$ between $H_{c1}$ and $H_{c2}$. Just above $H_{c1}$ (Fig. 6.6.4(a)), the vortices are far apart and have the shape described in Sec. 6.6 B — in particular the perturbations of $\Psi$ and $\mathbf{B}$ from their equilibrium values decays with $r$ to a negligible value well within the Wigner-Seitz cell. On the other hand, just before $H_{c2}$, the spacing between the vortex lines (defined by the places where $\Psi = 0$) is only a coherence length, which is much shorter than the penetration depth (Fig. 6.6.4(b)). 17 The Abrikosov lattice consists merely of small (almost sinusoidal) modulations around an average state which barely differs from the normal state: everywhere, the field $\mathbf{B}(\mathbf{r})$ is only slightly reduced from the external $H$, and the order parameter $|\Psi(\mathbf{r})|$ is never more than a small fraction of $\Phi_0$. (As it were, the single-vortex configuration of Fig. 6.6.3 gets truncated, before it even gets started, by the unit-cell boundary of Fig. 6.6.1.)

So it is not quite evident why the flux per unit cell $\Phi$ is exactly $\Phi_0^2$, even if the lattice constant is much smaller than the natural radius of a lone vortex. To see this, let first

---

17NMR can measure the profile of $B(\mathbf{r})$ as shown in Fig. 6.6.4.
6.6 W. LOWER CRITICAL FIELD $H_{c1}$

note that, by symmetry, the component of supercurrent $\mathbf{J}_s$ along the Wigner-Seitz cell edge vanishes. Thus the loop integral $\oint \mathbf{J}_s(\mathbf{r}) \cdot d\mathbf{r}$ along the cell boundary vanishes just as surely (though for different reasons) as the integral along a loop along the interior of a solid ring (see Sec. 6.4 C); from this it follows that $\Phi = n\Phi_0$. Since vortices repel, any case with $n > 2$ would split up into individual vortices, so $\Phi = \Phi_0$ as asserted.

Mathematical details of the estimation of $H_{c1}$ and $H_{c2}$ are left to Secs. 6.6 W and 6.6 X, where they are analyzed as instabilities of the superconducting and normal states, respectively. There are actually several phases of On closer examination (with statistical mechanics), the Abrikosov phase breaks up into several phases (Sec. 6.7 Y).

6.6 W Lower critical field $H_{c1}$

Now we turn to some details of computing the equation of state shown in Fig. 3(e). If we think of the vortices (in a 2D projection) as being particles, then the external magnetic field looks like a “pressure” term. The story of the vortices here is very analogous to that of the domain walls (discommensurations) in the Frenkel-Kontorova model of the commensurate-incommensurate transition (Lec. 3.3).

There is a certain field, above which the free energy for a single vortex to come in goes negative. That free energy, per unit length, is a combination of the vortex core energy and the “magnetic pressure” energy.

Beyond that field, the single-vortex energy tells us merely that the system is unstable to the entry of vortices, but does not tell us what is the stable state – i.e., how many vortices enter, or how they arrange themselves. We can, however, answer both those questions using the vortex-vortex effective interaction found in (6.6.21); that will suffice for finding how the ground state lattice constant $a$ is related to the external field.

**Useful conversions**

For use below, note the following handy manipulations (mostly repeated from Table 6.1.1).

Note that

$$H_c = \frac{\sqrt{2}}{4\pi} \left( \frac{\Phi_0}{\lambda_\xi} \right)$$

(6.6.27)

and so

$$|F_{\text{cond}}| = \frac{H_c^2}{8\pi} = \frac{1}{(4\pi)^2} \left( \frac{\Phi_0}{\lambda_\xi} \right)^2 .$$

(6.6.28)

**Details**

I showed in Sec. 6.3 A that given your sample has a field $H$ on the outside the effective magnetic energy density is

$$\frac{1}{8\pi} |\mathbf{B}(\mathbf{r}) - \mathbf{H}_{\text{app}}|^2 .$$

(6.6.29)

---

18This is easily checked when you write, as an approximation, the linear superposition of the current fields of the vortices. To show it generally, note that a mirror reflection in the vertical plane containing the cell edge reverses all the velocities (time-reversal) while on the other hand it leaves fixed the velocity component within the plane.

19This analogy is detailed in a figure and table which were included in 1999, edition but omitted in later years.

20Or in standard electromagnetic parlance, the magnetic induction is $H$ on the inside.

21This is completely analogous to the mismatch energy in the Frenkel-Kontorova model.
This term competes with the vortex-vortex repulsion (6.6.21).

*Sorry, I should write out the integral for the total free energy.* This integral can be rearranged in the same fashion as we did with the continuum model energy of the Frenkel-Kontorova model (Lec. 3.3). The result is that $H$ couples like a chemical potential (per unit length) favoring vortices, thus competing with the vortex core energy estimated in Sec. 6.6 C.

The resulting energy is

$$
(E_v - \frac{1}{4\pi} \Phi_0^* H) N_v Z + \text{const}
$$

where $E_v$ is the energy per unit length of the vortex, $N_v$ is the total number of vortices, and $Z$ is the size of the system in the direction the vortices run.

The transition at $H_{c1}$ is an instability of the vortex-free state to the entry of a few vortices. Thus, it occurs precisely when the coefficient in (6.6.30) goes negative:

$$
H_{c1} = 4\pi E_v / \Phi_0^*
$$

Now in Sec. 6.6 C we estimated that $E_v = CF_{\text{cond}} \xi^2$ for some prefactor $C$ of order unity (which probably contains a logarithm, but never mind), then $E_v = C H_c^2 / 8\pi$. We replace one factor of $H_c$ using the the relation

$$
H_c = \frac{\sqrt{2}}{4\pi} \left( \frac{\Phi_0^*}{\lambda\xi} \right)
$$

The end result of this manipulation is

$$
H_{c1} = \sqrt{2} C (\xi / \lambda) H_c = C \kappa^{-1} H_c
$$

as was claimed.

### 6.6 X Estimation of upper critical field $H_{c2}$

We ask about the instability of the normal state to developing a *small* $\Psi(r)$. (Recall Fig. 6.6.4(b).) We assume the field is along the $\hat{z}$ axis.

Notice that

(i). In this limit, $B(r) \simeq H$ and is virtually uniform within the superconductor. Thus we needn’t bother with the field energy (6.6.29); it’s just a constant. (See Fig. 6.6.4.)

(ii). Since the order parameter $\Psi(r)$ is small everywhere when $H$ is near $H_{c2}$, it will be OK to linearize; we needn’t bother with the $\frac{1}{2} \beta |\Psi(r)|^2$ term in the Landau free energy.

Thanks to (i) and (ii), our total free energy is simply a *quadratic form* in $\Psi(r)$:

$$
F_{\text{tot}} \simeq \int d^3 r \left\{ \frac{\hbar^2}{2m^*} \left| \nabla - \frac{ie}{\hbar c} A(r) \Psi(r) \right|^2 + \alpha |\Psi(r)|^2 \right\}
$$

(Here $\nabla \times A = B = H$. Recall $\alpha = -|\alpha|$, since certainly $T < T_c(0)$, where $T_c(0)$ is critical temperature in zero field.

\footnote{These handy identities are from Table 6.1.1.}
In principle, then, we should completely diagonalize the quadratic form (6.6.34). At \( B > H_{c2} \), every eigenvalue will be positive, and \( H_{c2} \) will be defined as the largest value of \( B \) at which any eigenvalue crosses through zero and becomes negative. Now, if \( \Psi(\mathbf{r}) \) were an ordinary (normalized) single-particle wavefunction, then (6.6.34) would be simply the energy expectation \( \langle \mathcal{H} \rangle \) of a certain Hamiltonian

\[
\mathcal{H} = -\frac{\hbar^2}{2m_\ast} \left( \nabla - \frac{ie_\ast}{\hbar c} \mathbf{A}(\mathbf{r}) \right)^2 - |\alpha| \tag{6.6.35}
\]

We want to see where the lowest eigenvalue \( E_0 \) crosses zero; this is just the ground state eigenvalue \( E_0 \) of \( \mathcal{H}(\psi) = E_0|\psi\rangle \).

But – apart from the constant term added at the end – (6.6.35) is simply the Hamiltonian for a free charged particle in a magnetic field \( H \). The solution is thus a Landau Level wavefunction (see Lec. 9.1 [omitted]). The quantum numbers are \( k_z \), the wavevector in the direction of the field, and \( n \), the Landau level index. The energy is

\[
E(n, k_z) = (n + \frac{1}{2})\hbar \omega_c + \frac{\hbar^2}{2m_\ast} k_z^2 - |\alpha|. \tag{6.6.36}
\]

where

\[
\omega_c(H) \equiv e_\ast H/m_\ast c \tag{6.6.37}
\]

would be the cyclotron frequency of the free charged particle. The minimum energy is \( E(0, 0) \). The critical field is where this passes through zero, i.e. it is given by

\[
\frac{1}{2} \omega_c(H_{c2}) = |\alpha| \tag{6.6.38}
\]

Inserting the definition (6.6.37) and rewriting \( |\alpha| \), (6.6.38) becomes \( \frac{1}{2} \hbar (e^* H_{c2} / m^* c) = \hbar^2 / 2m^* \xi^2 \), or

\[
H_{c2} = \frac{1}{2\pi} \frac{\Phi_0^*}{\xi^2} \tag{6.6.39}
\]

where I inserted the definition of the flux quantum \( \Phi_0^* = \hbar c / e^* \). From (6.6.39), it’s clear that the unit cell of the vortex lattice is indeed \( \sim \xi \) at \( H_{c2} \). Finally (using \( H_c \equiv \frac{\Phi_0^*}{2\pi \xi^2} \)) we find

\[
H_{c2} = \sqrt{2} \frac{\lambda}{\xi} H_c \tag{6.6.40}
\]

as in (6.6.24).

What this section has done is, conceptually, just like the Landau theory, within which the critical point was \textit{exactly} the place where the coefficient of the quadratic form \( F_L(\Phi) \) went negative. Once that coefficient was negative, the state \( \Phi = 0 \) became unstable; if we are curious to \textit{what} state (e.g. what value of \( \Phi \)) is unstable, we must work harder (e.g. must take account of and understand the nonlinear terms). (The present case was a bit more complicated than Landau theory because the instability occurs to a spatially modulated state, not a uniform one.)

\section*{Exercises}

\textbf{Ex. 6.6.1 Vortex interaction in Type I? (T)}

Thought question: do two vortex lines attract, or repel, in a Type I superconductor?

\footnote{
That was discussed in Lec. 1.4 C.
}
**Ex. 6.6.2 Energy of multiply charged vortex (T)**

This question is about a Type II superconductor, or a neutral superfluid.

(a). In the case of a general vortex charge \( n \), how does the gradient energy in (6.6.13) scale with \( n \)?

(b). If a single vortex with \( n = 2 \) breaks up into two vortices, each with \( n = 1 \), by what factor is the total cost changed?

(c). Is this what you would have guessed from the negative domain wall energy \( \sigma_W \) in Type II?

**Ex. 6.6.3 Variational calculation for vortex**

The purpose of this exercise is to check the statements about the radial dependence and line energy of a vortex, using the crudest possible variational form.

Try

\[
\Psi(r, \theta, z) = \Psi_0 e^{i\theta} \times \begin{cases} (r/a) & 0 < r < a; \\ 1 & r > a. \end{cases}
\]

\[
A(r, \theta, z) = \frac{\Phi_0}{2\pi} \times \begin{cases} r/b^2 & 0 < r < b; \\ (1/r) & r > b. \end{cases}
\]

(6.6.41)

(6.6.42)

(The coefficients must be the values given in order that the energy density goes to zero far from the vortex’s axis; then (6.6.42) ensures the flux is the flux quantum \( \Phi_0^* \). The second crudest form would have radial dependences \( e^{-r/a} \) and \((1 - e^{-r/b})/r\), respectively.)

(a) Compute each of the three terms in the total G-L free energy difference (compared to the uniform state \( \Psi = \Psi_0 \)), in the form of \( F_{\text{cond}} L_z \) times a function of \( a, b, \xi, \lambda \). (The sooner you can eliminate \( a, \beta, m_\phi \) in favor of the G-L length scales, the better.)

Some hints (you should still work them, not assume them) (i) \( F_L = F_{\text{cond}} \left( 1 - |\Psi|^2/|\Psi_0|^2 \right) \). (ii) \( F_{\text{grad}} \) has a factor \((1 - r - r/b^2)^2 \) both in the \((0, a)\) and in the \((a, b)\) ranges. (iii) \( \xi \) should appear only in the \( F_{\text{grad}} \) term, and \( \lambda \) only in the magnetic field energy term.

(b) Then, write the equations to minimize it with respect to \( a \) and \( b \). Intuition (and the assertions in the text) suggest that \( a \propto \xi \) and \( b \propto \lambda \). If necessary to simplify the result, you may assume \( \xi \ll \lambda \) (extreme type II case).

**Ex. 6.6.4 Anisotropic case with vortex**

Imagine you have different effective masses \( m_x \) and \( m_y \) in the \( x \) and \( y \) directions.

(b) Show you can rescale the system with variables \( x' = \gamma x, y' = y/gamma \) so that every term in the G-L free energy looks isotropic again. (Notice that you must rescale \( A_x \) and \( A_y \) correspondingly so that \( \int r A \cdot d \ll - \) the flux – is unchanged.)

(b) The isotropic G-L system always has a triangular vortex lattice, which can be considered centered rectangular whose lattice constants have a ratio \( b/a = \sqrt{3} \). What is the lattice for the anisotropic case?